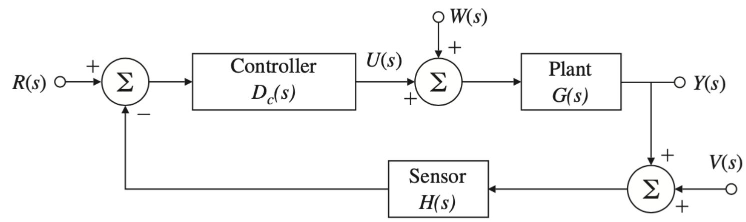
Solutions of Final Exam

Subject : Control System Engineering 1, Lecturer : Prof. Youngjin Choi,
Date : June 19, 2020 (Contact e-mail : cyj@hanyang.ac.kr)

Problem 1 (20pt) Consider the following figure with $G(s) = \frac{1}{s(\tau s+1)}$, $D_c(s) = k_P + k_D s$ and $H(s) = 1 + k_t s$. Determine the system type and relevant error constant with respect to the reference inputs when $V = W = 0$.



Solution of Problem 1 (20pt)
The error TF is

$$\begin{aligned} E(s) &= R(s) - \frac{D_c G}{1 + D_c G H} R(s) \\ &= \left[1 - \frac{\frac{k_P + k_D s}{s(\tau s+1)}}{1 + \frac{(k_P + k_D s)(1+k_t s)}{s(\tau s+1)}} \right] R(s) \\ &= \left[1 - \frac{k_P + k_D s}{s(\tau s + 1) + (k_P + k_D s)(1 + k_t s)} \right] R(s) \\ &= \left[1 - \frac{k_P + k_D s}{(\tau + k_t k_D)s^2 + (k_P k_t + k_D + 1)s + k_P} \right] R(s) \\ &= \frac{(\tau + k_t k_D)s^2 + (k_P k_t + 1)s}{(\tau + k_t k_D)s^2 + (k_P k_t + k_D + 1)s + k_P} R(s) \end{aligned}$$

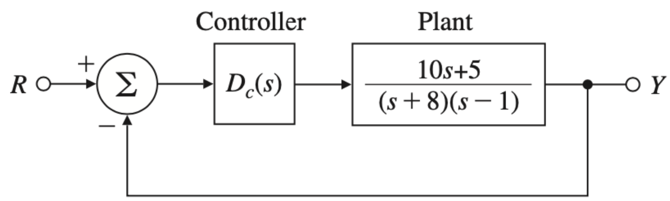
The steady-state error for inputs (step, ramp, parabolic) becomes

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s \frac{s[(\tau + k_t k_D)s + (k_P k_t + 1)]}{(\tau + k_t k_D)s^2 + (k_P k_t + k_D + 1)s + k_P} \frac{1}{s} = 0 \\ e_{ss} &= \lim_{s\to 0} s \frac{s[(\tau + k_t k_D)s + (k_P k_t + 1)]}{(\tau + k_t k_D)s^2 + (k_P k_t + k_D + 1)s + k_P} \frac{1}{s^2} = \frac{k_P k_t + 1}{k_P} \qquad \text{type 1} \\ e_{ss} &= \lim_{s\to 0} s \frac{s[(\tau + k_t k_D)s + (k_P k_t + 1)]}{(\tau + k_t k_D)s^2 + (k_P k_t + k_D + 1)s + k_P} \frac{1}{s^3} = \infty \end{aligned}$$

Thus, the "system type is 1" and its velocity error constant becomes

$$K_v = \frac{k_P}{k_P k_t + 1}$$

Problem 2 (20pt) Consider the controller of the form $D_c(s) = \frac{1}{s^n}$ with n being a non-negative integer. For what values of n is the closed-loop system stable?



Solution of Problem 2 (20pt) The characteristic equation becomes

$$1 + \frac{1}{s^n}\frac{10s+5}{(s+8)(s-1)} = 0 \quad \rightarrow \quad s^n(s^2+7s-8)+10s+5=0$$

If $n=0$, the necessary condition is not satisfied

$$s^2+17s-3=0 \quad \text{unstable}$$

If $n=1$, the necessary condition is satisfied

$$s^3+7s^2+2s+5=0$$

Let us take Routh Table to check the sufficient condition

$$\begin{array}{lll} s^3: & 1 & 2 \\ s^2: & 7 & 5 \\ s^1: & \frac{9}{7} & \\ s^0: & 5 & \end{array}$$

Since the sufficient condition is satisfied, the system when $n=1$ is stable

If $n=2$, the necessary condition is not satisfied

$$s^4+7s^3-8s^2+10s+5=0 \quad \text{unstable}$$

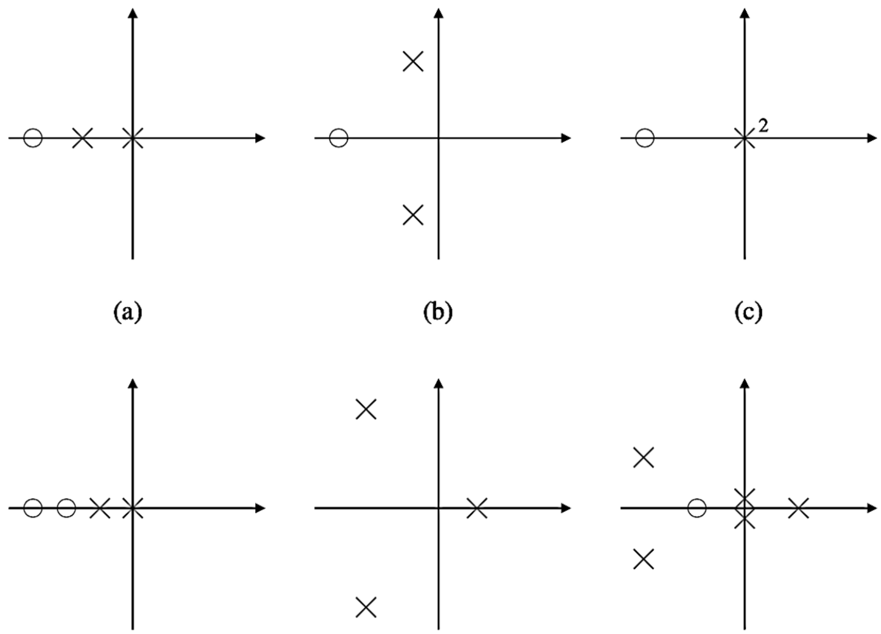
If $n \geq 3$, the necessary condition is not satisfied

$$s^{n+2}+7s^{n+1}-8s^n+\cdots+0s^2+10s+5=0 \quad \text{unstable}$$

Problem 3 (20pt) Roughly sketch the root loci for the following pole-zero maps. Each pole-zero map is from a characteristic equation of the form:

$$1 + K\frac{b(s)}{a(s)} = 0$$

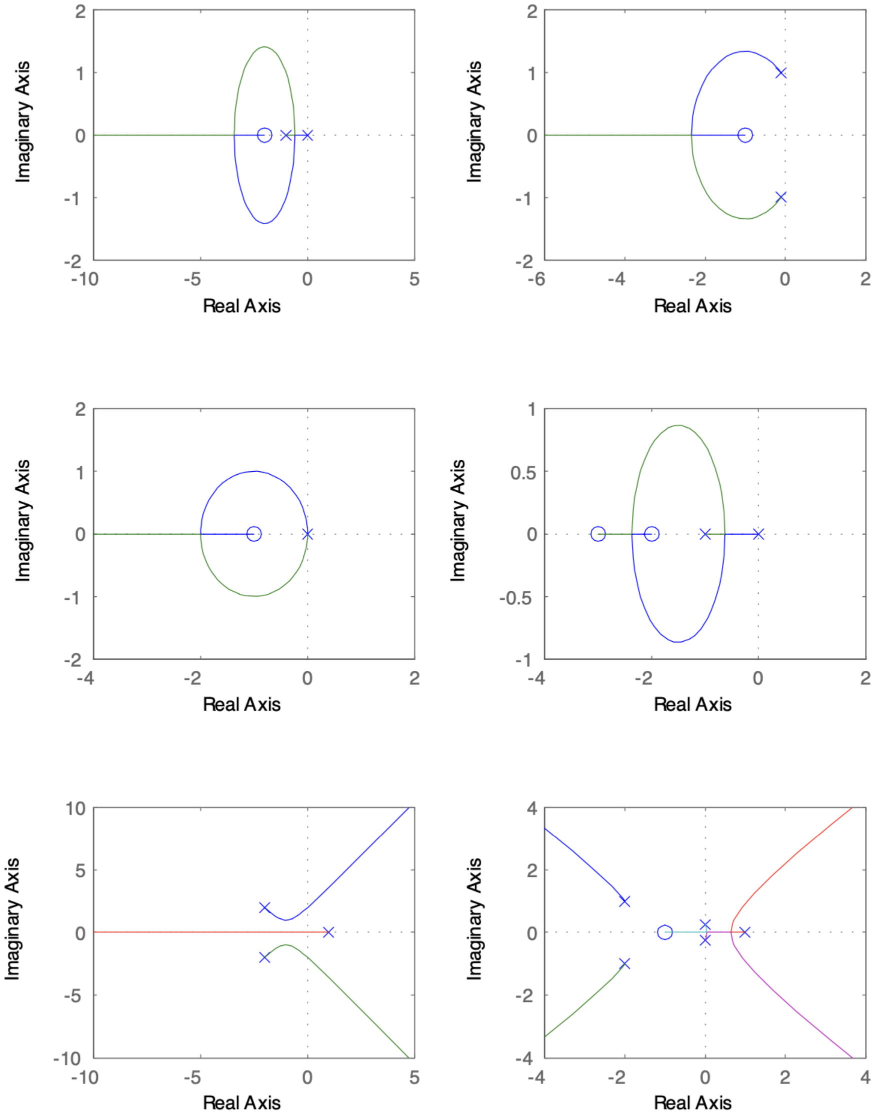
where the roots of the $b(s)$ are shown as small circles $\circ$ and the roots of the $a(s)$ are shown as $\times$ on the s-plane. Note, in figure (c), there are two poles at the origin.



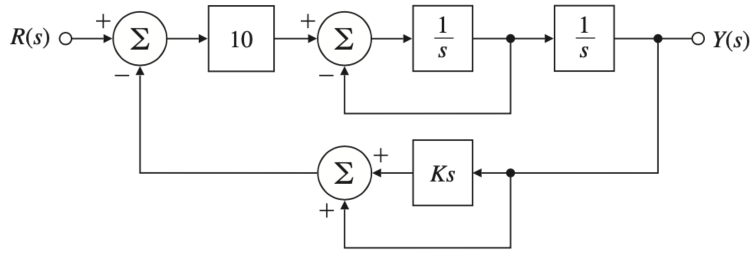
(a) (b) (c)

(d) (e) (f)

Solution of Problem 3 (20pt)



Problem 4 (20pt) For the feedback system shown in the figure, find the value of the gain K that results in dominant closed-loop poles with a damping ratio $\zeta = \frac{1}{\sqrt{2}}$



Solution of Problem 4 (20pt) The transfer function becomes

$$\frac{Y(s)}{R(s)} = \frac{\frac{10}{s(s+1)}}{1 + \frac{10(1+Ks)}{s(s+1)}}$$
$$= \frac{10}{s^2 + (10K+1)s + 10}$$

From the characteristic equation, we know

$$\omega_n = \sqrt{10} \qquad 2\zeta\omega_n = 10K + 1$$
$$2\sqrt{5} = 10K + 1 \quad \rightarrow \quad K = 0.347$$

Problem 5 (20pt) Draw the root locus with respect to K for the equation $1 + KL(s) = 0$, where the departure angles should be calculated and suggested.

$$L(s) = \frac{s+3}{s(s+10)(s^2+2s+2)}$$

Solution of Problem 5 (20pt)

1. (Rule 1, Start and End)

$$\begin{aligned} &\text{when } K = 0, \quad s = 0, -10, -1 \pm j, \quad \text{poles of } L(s) = \tfrac{s+3}{s(s+10)(s^2+2s+2)} \\ &\text{when } K = \infty, \quad s = -3, \infty, \infty, \infty \quad \text{zeros of } L(s) \end{aligned}$$

2. (Rule 2, Real Axis) Negative real axis of $-10 \le s$ and $-3 \le s \le 0$ are loci

3. (Rule 3, Asymptotes)

$$\phi_l = \frac{180^\circ + 360(l-1)}{4-1} = \pm 60^\circ, \pm 180^\circ \qquad \alpha = \frac{(-12) - (-3)}{3} = -3$$

4. (Rule 4, Departure Angles and Arrival Angles)

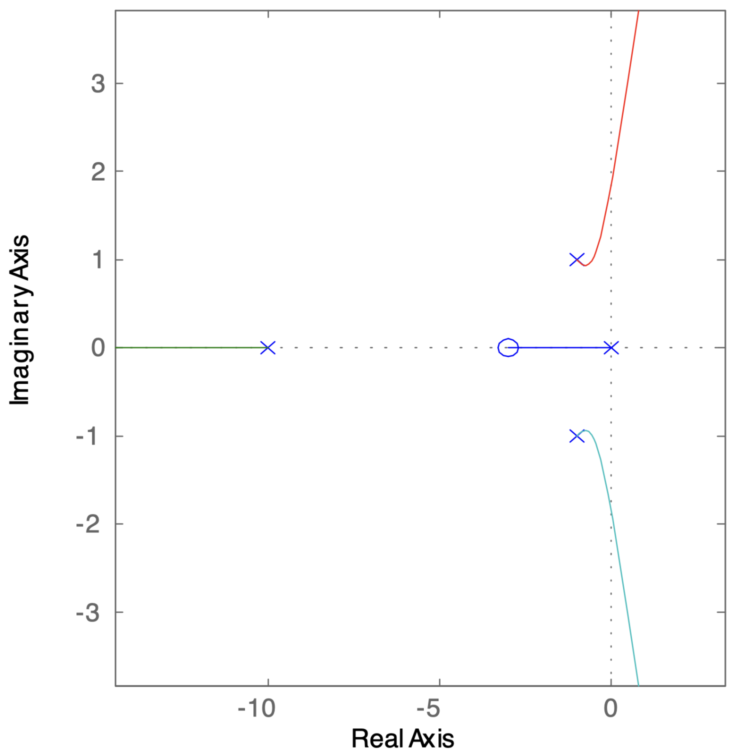
$$\begin{aligned} \phi_{dep,-1+j} &= \angle(-1+j-(-3)) - \angle(-1+j-0) - \angle(-1+j-(-1-j)) - \angle(-1+j-(-10)) - 180^\circ \\ &= \tan^{-1}\frac{1}{2} - (180^\circ - \tan^{-1} 1) - 90^\circ - \tan^{-1}\frac{1}{9} - 180^\circ \\ &= 26.5^\circ - (180^\circ - 45^\circ) - 90^\circ - 6.3^\circ - 180^\circ = -24.8^\circ \\ \phi_{dep,-0.1-6.6j} &= 24.8^\circ \end{aligned}$$

5. (Rule 5, Break-in and Breakaway Points) No break-in and breakaway points.

6. Routh table from the characteristic equation : $s^4 + 12s^3 + 22s^2 + (20+K)s + 3K = 0$

$$\begin{array}{lll} s^4 : 1 & 22 & 3K \\ s^3 : 12 & 20+K & \\ s^2 : 22 - \dfrac{20+K}{12} & 3K & \\ s^1 : 20 + K - \dfrac{36K}{22 - \frac{20+K}{12}} & & \\ s^0 : 3K & & \end{array}$$

For $0 < K < 23$, the stability is guaranteed, but at $K = 23$, the roots pass through the imaginary axis at $\omega = \pm 1.94j$


3
2
1
0
-1
-2
-3
-10
-5
0
Real Axis
Imaginary Axis