# Solutions of Midterm Exam

Subject : Modern Robotics, Lecturer : Prof. Youngjin Choi,
Date : May 14, 2020 (Contact e-mail : cyj@hanyang.ac.kr)
Notice that the answers should be written only in English, otherwise you will get a zero point.

Problem 1 (20pt) The following figure shows a table lamp that moves only in the plane of the page. Use Grübler's formula to calculate the number of degrees-of-freedom.

Solution of Problem 1 (20pt) Despite all the links and revolute joints, this mechanical system behaves similarly to a 3R robot arm, since each set of two revolute joints acts as a single hinge.

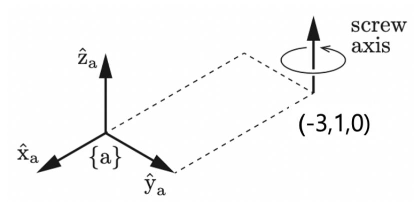
Consider a mechanism consisting of $N = 8$ links, where ground is regarded as a link. Let

- $J = 9$ due to three overlapped revolute joints
- $m = 3$ for planar mechanisms
- $f_i = 1$ because all the joints are revolute

Then Grübler's formula for the number of dof of the robot is

$$\begin{aligned} \text{dof} &= m(N-1-J) + \sum_{i=1}^{J} f_i \\ &= 3(8-1-9)+9 \\ &= 3 \end{aligned}$$

Problem 2 (20pt) The zero-pitch screw axis in the following figure, aligned with $\hat{z}_a$, passes through the point $(-3, 1, 0)$ in the {a} frame. What is the twist $\mathcal{V}_a$ if we rotate about the screw axis at a speed $\dot{\theta} = 10[rad/s]$?




Solution of Problem 2 (20pt) Since the rotational axis is aligned with $\hat{z}_a$, we have

$$\hat{\omega}_a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From the figure and simple geometry $q_a = (-3, 1, 0)$, we get

$$\begin{aligned} v_a &= \hat{\omega}_a \times (-q_a) = [\hat{\omega}_a](-q_a) \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \end{aligned}$$

and thus we obtain the screw axis $\mathcal{S}_a$ and twist $\mathcal{V}_a$:

$$\mathcal{S}_a = \begin{bmatrix} \hat{\omega}_a \\ v_a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \qquad \mathcal{V}_a = \mathcal{S}_a \dot{\theta} = \begin{bmatrix} 0 \\ 0 \\ 10 \\ 10 \\ 30 \\ 0 \end{bmatrix}$$

Problem 3 (20pt) The RRRP SCARA robot of the following figure is shown in its zero position. For $l_0 = l_1 = l_2 = 1$, determine the end-effector zero position configuration $M$ and the screw axes $\mathcal{S}_i$ in $\{0\}$. Find the end-effector configuration $T_{0b} \in SE(3)$ when $\theta = (0, \frac{\pi}{2}, 0, 1)$.

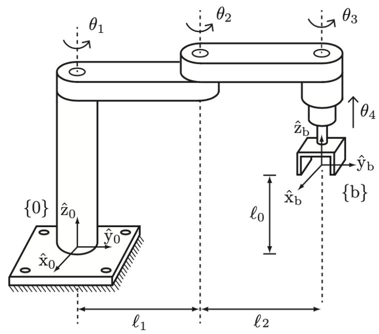



**Figure 4.12:** An RRRP SCARA robot for performing pick-and-place operations.

Solution of Problem 3 (20pt) The end-effector zero position configuration is

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

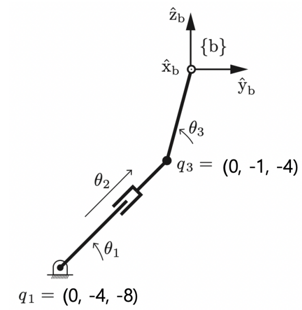
and the screw axes $\mathcal{S}_i = (\hat{\omega}_i, \hat{\omega}_i \times (q_i))$ for $i = 1, 2, 3, 4$ are

$$\mathcal{S}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{S}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{S}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The end-effector configuration $T_{04} \in SE(3)$ when $\theta = (0, \frac{\pi}{2}, 0, 1)$ becomes

$$\begin{aligned} T_{0b} &= e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} M \\ &= e^0 e^{[\mathcal{S}_2]\frac{\pi}{2}} e^0 e^{[\mathcal{S}_4](1)} M \\ &= \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Problem 4 (20pt) The following figure shows an RPR robot that is confined to the plane of the page. An end-effector frame {b} is illustrated, where the $\hat{x}_b$-axis is out of the page. The directions of positive motion of the three joints are indicated by arrows. The axes of the two revolute joints are out of the page, and the prismatic joint moves in the plane of the page. Joint 1 is at $q_1 = (0, -4, -8)$ in {b} and joint 3 is at $q_3 = (0, -1, -4)$ in {b}. Write the body Jacobian $J_b(\theta)$ for the configuration shown. All entries of your $J_b(\theta)$ matrix should be numerical (no symbols or math).




Solution of Problem 4 (20pt)

You can see this by visualization (imagine turntables at joints 1 and 3 and visualize the motion of a point at the origin of {b}, and imagine a conveyor moving in the direction of joint 2) or by recognizing that $\omega_1 = \omega_3 = (1, 0, 0)$ and points on the joint 1 and 3 axes are $q_1$ and $q_3$ and calculating $v_i = \omega_i \times (-q_i)$ for $i = 1, 3$. For joint 2, the linear direction of positive motion is given by $v_2 = \frac{(q_3 - q_1)}{\|q_3 - q_1\|}$. The screw axes at the given configuration are

$$\mathcal{B}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -8 \\ 4 \end{bmatrix} \qquad \mathcal{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \qquad \mathcal{B}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Therefore,

$$J_b(\theta) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -8 & \frac{3}{5} & -4 \\ 4 & \frac{4}{5} & 1 \end{bmatrix}$$

Problem 5 (20pt) Find the joint velocities $\dot{\theta}$ minimizing the following index with $W = W^T > 0$.

$$\min\ \frac{1}{2}\dot{\theta}^T W \dot{\theta} + \dot{\theta}^T g(\theta) \qquad \text{subject to } \mathcal{V}_d = J\dot{\theta}$$

Solution of Problem 5 (20pt)

$$\begin{aligned}
H &= \frac{1}{2}\dot{\theta}^T W \dot{\theta} + \dot{\theta}^T g(\theta) + \lambda^T(\mathcal{V}_d - J\dot{\theta}) \\
\frac{\partial H}{\partial \dot{\theta}} &= W\dot{\theta} + g(\theta) - J^T\lambda = 0 \\
\mathcal{V}_d &= J\dot{\theta} = JW^{-1}[J^T\lambda - g(\theta)] \\
\lambda &= (JW^{-1}J^T)^{-1}[\mathcal{V}_d + JW^{-1}g(\theta)] \\
\dot{\theta} &= W^{-1}J^T\lambda - W^{-1}g(\theta) \\
&= W^{-1}J^T(JW^{-1}J^T)^{-1}\mathcal{V}_d + W^{-1}J^T(JW^{-1}J^T)^{-1}JW^{-1}g(\theta) - W^{-1}g(\theta) \\
&= J_W^+\mathcal{V}_d + (I - J_W^+ J)W^{-1}(-g(\theta))
\end{aligned}$$

where $J_W^+ = W^{-1}J^T(JW^{-1}J^T)^{-1}$