1 Manipulator Jacobian

- In the most general case, v_{tip} can be taken to be a six-dimensional twist $\mathcal{V}$, while, for pure orienting devices such as a wrist, v_{tip} is usually taken to be the angular velocity of the end-effector frame.

$$\mathcal{V} = J(\theta)\dot{\theta} \quad \rightarrow \quad \mathcal{V} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2 + \cdots + J_n(\theta)\dot{\theta}_n$$

  For given the configuration θ of the robot, the 6-vector $J_i(\theta)$, which is column i of $J(\theta)$, is the twist $\mathcal{V}$ when $\dot{\theta}_i = 1$ and all other joint velocities are zero.

- The only difference is that the screw axes of the Jacobian depend on the joint variables θ whereas the screw axes for the forward kinematics of Chapter 4 were always for the case $\theta = 0$.

$$J_i(\theta) = \text{screw axis for arbirary } \theta$$

- The two standard types of Jacobian that we will consider are
  - space Jacobian $J_s(\theta)$ satisfying

$$\mathcal{V}_s = J_s(\theta)\dot{\theta} = J_{s1}\dot{\theta}_1 + J_{s2}\dot{\theta}_2 + \cdots + J_{sn}\dot{\theta}_n$$

    where each column $J_{si}(\theta)$ corresponds to a screw axis expressed in the fixed space frame {s}
  - body Jacobian $J_b(\theta)$ satisfying

$$\mathcal{V}_b = J_b(\theta)\dot{\theta} = J_{b1}\dot{\theta}_1 + J_{b2}\dot{\theta}_2 + \cdots + J_{bn}\dot{\theta}_n$$

    where each column $J_{bi}(\theta)$ corresponds to a screw axis expressed in the end-effector frame {b}.

1.1 Space Jacobian

Consider an n-link open chain whose forward kinematics is expressed in the following PoE form:

$$T(\theta_1, \cdots, \theta_n) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M$$

Using the following properties

$$\begin{aligned}
\dot{T} &= \frac{de^{[\mathcal{S}_1]\theta_1}}{dt} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M + e^{[\mathcal{S}_1]\theta_1} \frac{de^{[\mathcal{S}_2]\theta_2}}{dt} \cdots e^{[\mathcal{S}_n]\theta_n} M + \cdots + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots \frac{de^{[\mathcal{S}_n]\theta_n}}{dt} M \\
&= [\mathcal{S}_1]\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M + e^{[\mathcal{S}_1]\theta_1} [\mathcal{S}_2]\dot{\theta}_2 e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M + \cdots + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots [\mathcal{S}_n]\dot{\theta}_n e^{[\mathcal{S}_n]\theta_n} M \\
T^{-1} &= M^{-1} e^{-[\mathcal{S}_n]\theta_n} e^{-[\mathcal{S}_{n-1}]\theta_{n-1}} \cdots e^{-[\mathcal{S}_1]\theta_1}
\end{aligned}$$

the spatial twist $\mathcal{V}_s \in \Re^6$ and its matrix form $[\mathcal{V}_s] = \dot{T}T^{-1} \in se(3)$ are obtained as:

$$\begin{aligned}
[\mathcal{V}_s] &= \dot{T}T^{-1} \\
&= [\mathcal{S}_1]\dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1} [\mathcal{S}_2] e^{-[\mathcal{S}_1]\theta_1} \dot{\theta}_2 + e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} [\mathcal{S}_3] e^{-[\mathcal{S}_2]\theta_2} e^{-[\mathcal{S}_1]\theta_1} \dot{\theta}_3 + \cdots
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}_s &= \mathcal{S}_1\dot{\theta}_1 + Ad_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)\dot{\theta}_2 + Ad_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)\dot{\theta}_3 + \cdots \\
&= J_{s1}\dot{\theta}_1 + J_{s2}(\theta_1)\dot{\theta}_2 + J_{s3}(\theta_1, \theta_2)\dot{\theta}_3 + \cdots = J_s(\theta)\dot{\theta}
\end{aligned}$$

where it is noted that ith column of Jacobian corresponds to the adjoint mapping of ith screw axis

$$J_{si}(\theta) = Ad_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i)$$

Definition 5.1. *Let the FK of an n-link open chain be expressed in the following PoE form:*

$$T(\theta_1, \cdots, \theta_n) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M \in SE(3)$$

The space Jacobian $J_s(\theta) \in \Re^{n\times 6}$ relates the joint rate vector $\dot{\theta} \in \Re^n$ to the spatial twist $\mathcal{V}_s$ via

$$\mathcal{V}_s = J_s(\theta)\dot{\theta}$$

The ith column of $J_s(\theta)$ is

$$J_{si}(\theta) = Ad_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i)$$

for $i = 2, \cdots, n$, with the first column $J_{s1} = \mathcal{S}_1$.

- To understand the physical meaning behind the columns of $J_s(\theta)$,
  - observe that the ith column is of the form $Ad_{T_{i-1}}(\mathcal{S}_i)$, where $T_{i-1} = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}$
  - recall that $\mathcal{S}_i$ is the screw axis describing the ith joint axis in terms of the fixed frame with the robot in its zero position.
- $Ad_{T_{i-1}}(\mathcal{S}_i)$ is therefore the screw axis describing the ith joint axis after it undergoes the rigid body displacement $T_{i-1} = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}$.
- The procedure for determining the columns J_{si} of $J_s(\theta)$ is similar to the procedure for deriving the joint screws $\mathcal{S}_i$ in the PoE formula $e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M$: each column $J_{si}(\theta)$ is the screw vector describing joint axis i, expressed in fixed-frame coordinates, but for arbitrary θ rather than $\theta = 0$.

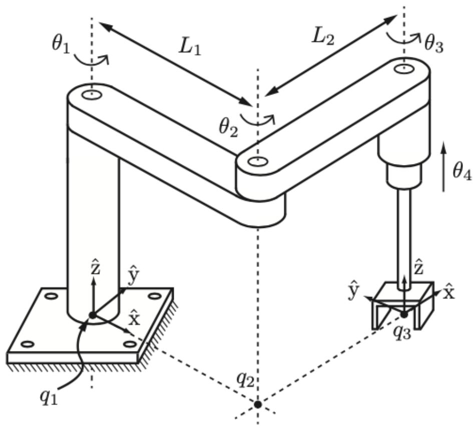

Figure 5.7: Space Jacobian for a spatial RRRP chain.

Example 5.1. *Space Jacobian* $J_s(\theta)$ *for a spatial RRRP chain with* $J_{si} = [\omega_{si}, v_{si}]$ *?*

- $\omega_{s1} = [0, 0, 1]$, $q_1 = [0, 0, 0]$, **and** $v_{s1} = [0, 0, 0]$
- $\omega_{s2} = [0, 0, 1]$, $q_2 = [L_1 c_1, L_1 s_1, 0]$, **and** $v_{s2} = \omega_{s2} \times (-q_2) = [L_1 s_1, -L_1 c_1, 0]$
- $\omega_{s3} = [0, 0, 1]$, $q_3 = [L_1 c_1 + L_2 c_{12}, L_1 s_1 + L_2 s_{12}, 0]$, **and** $v_{s3} = \omega_{s3} \times (-q_3) = [L_1 s_1 + L_2 s_{12}, -L_1 c_1 - L_2 c_{12}, 0]$
- $\omega_{s4} = [0, 0, 0]$, **and** $v_{s4} = [0, 0, 1]$

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & L_1 s_1 & L_1 s_1 + L_2 s_{12} & 0 \\ 0 & -L_1 c_1 & -L_1 c_1 - L_2 c_{12} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $c_1 = \cos\theta_1$, $s_1 = \sin\theta_1$, $c_{12} = \cos(\theta_1 + \theta_2)$ **and** $s_{12} = \sin(\theta_1 + \theta_2)$

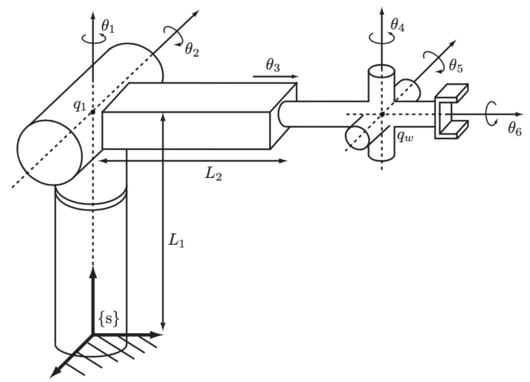

Figure 5.8: Space Jacobian for the spatial RRPRRR chain.

Example 5.2. *Space Jacobian for a spatial RRPRRR chain?*

- $\omega_{s1} = [0,0,1]^T$, $q_1 = [0,0,L_1]^T$, and $v_{s1} = \omega_{s1} \times (-q_1) = [0,0,0]^T$
- $\omega_{s2} = Rot(\hat{z},\theta_1)[-1,0,0]^T = [-c_1,-s_1,0]^T$, $q_2 = [0,0,L_1]^T$, and $v_{s2} = \omega_{s2} \times (-q_2) = [L_1 s_1, -L_1 c_1, 0]^T$
- $\omega_{s3} = [0,0,0]^T$, and $v_{s3} = Rot(\hat{z},\theta_1)Rot(\hat{x},-\theta_2)[0,1,0]^T = [-s_1 c_2, c_1 c_2, -s_2]^T$
- The wrist center is located at the point

$$q_w = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + Rot(\hat{z},\theta_1)Rot(\hat{x},-\theta_2)\begin{bmatrix} 0 \\ L_2+\theta_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -(L_2+\theta_3)s_1 c_2 \\ (L_2+\theta_3)c_1 c_2 \\ L_1-(L_2+\theta_3)s_2 \end{bmatrix}$$

- $\omega_{s4} = Rot(\hat{z},\theta_1)Rot(\hat{x},-\theta_2)[0,0,1]^T = [-s_1 s_2, c_1 s_2, c_2]^T$ and $v_{s4} = \omega_{s4} \times (-q_w)$
- $\omega_{s5} = Rot(\hat{z},\theta_1)Rot(\hat{x},-\theta_2)Rot(\hat{z},\theta_4)[-1,0,0]^T$ and $v_{s5} = \omega_{s5} \times (-q_w)$
- $\omega_{s6} = Rot(\hat{z},\theta_1)Rot(\hat{x},-\theta_2)Rot(\hat{z},\theta_4)Rot(\hat{x},-\theta_5)[0,1,0]^T$ and $v_{s6} = \omega_{s6} \times (-q_w)$

1.2 Body Jacobian

Consider a body form for the FK of n-link open chain

$$T(\theta_1, \cdots, \theta_n) = Me^{[\mathcal{B}_1]\theta_1}e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}$$

Using the following properties

$$\begin{aligned}
\dot{T} &= M\frac{de^{[\mathcal{B}_1]\theta_1}}{dt}e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n} + \cdots + Me^{[\mathcal{B}_1]\theta_1}e^{[\mathcal{B}_2]\theta_2}\cdots\frac{de^{[\mathcal{B}_n]\theta_n}}{dt} \\
&= Me^{[\mathcal{B}_1]\theta_1}[\mathcal{B}_1]\dot{\theta}_1e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n} + \cdots + Me^{[\mathcal{B}_1]\theta_1}e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}[\mathcal{B}_n]\dot{\theta}_n \\
T^{-1} &= e^{-[\mathcal{B}_n]\theta_n}e^{-[\mathcal{B}_{n-1}]\theta_{n-1}}\cdots e^{-[\mathcal{B}_1]\theta_1}M^{-1}
\end{aligned}$$

the body twist $\mathcal{V}_b \in \Re^6$ and its matrix form $[\mathcal{V}_b] = T^{-1}\dot{T} \in se(3)$ are obtained as:

$$\begin{aligned}
[\mathcal{V}_b] &= T^{-1}\dot{T} \\
&= e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_2]\theta_2}[\mathcal{B}_1]e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_1 + \cdots + e^{-[\mathcal{B}_n]\theta_n}[\mathcal{B}_{n-1}]e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1} + [\mathcal{B}_n]\dot{\theta}_n
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}_b &= Ad_{e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_2]\theta_2}}(\mathcal{B}_1)\dot{\theta}_1 + \cdots + Ad_{e^{-[\mathcal{B}_n]\theta_n}}(\mathcal{B}_{n-1})\dot{\theta}_{n-1} + \mathcal{B}_n\dot{\theta}_n \\
&= J_{b1}(\theta_2, \cdots, \theta_n)\dot{\theta}_1 + \cdots + J_{b,n-1}(\theta_n)\dot{\theta}_{n-1} + J_{bn}\dot{\theta}_n = J_b(\theta)\dot{\theta}
\end{aligned}$$

where it is noted that ith column of Jacobian corresponds to the adjoint mapping of ith screw axis

$$J_{bi}(\theta) = Ad_{e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i)$$

Definition 5.2. *Let the FK of an n-link open chain be expressed in the following PoE form:*

$$T(\theta_1, \cdots, \theta_n) = Me^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} \cdots e^{[\mathcal{B}_n]\theta_n} \in SE(3)$$

The body Jacobian $J_b(\theta) \in \Re^{n \times 6}$ relates the joint rate vector $\dot{\theta} \in \Re^n$ to the body twist $\mathcal{V}_b$ via

$$\mathcal{V}_b = J_b(\theta)\dot{\theta}$$

The ith column of $J_b(\theta)$ is

$$J_{bi}(\theta) = Ad_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i)$$

for $i = 1, \cdots, n-1$, with the last column $J_{bn} = \mathcal{B}_n$.

- Each column $J_{bi}(\theta) = (\omega_{bi}(\theta), v_{bi}(\theta))$ of $J_b(\theta)$ is the screw vector for joint axis i, expressed in the coordinates of the end-effector frame rather than those of the fixed frame.
- The procedure for determining the columns of $J_b(\theta)$ is similar to the procedure for deriving the forward kinematics in the PoE form $Me^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} \cdots e^{[\mathcal{B}_n]\theta_n}$
- Each of the end-effector-frame joint screws $J_{bi}(\theta)$ is expressed for arbitrary θ rather than $\theta = 0$.

1.3 Visualizing the Space and Body Jacobian

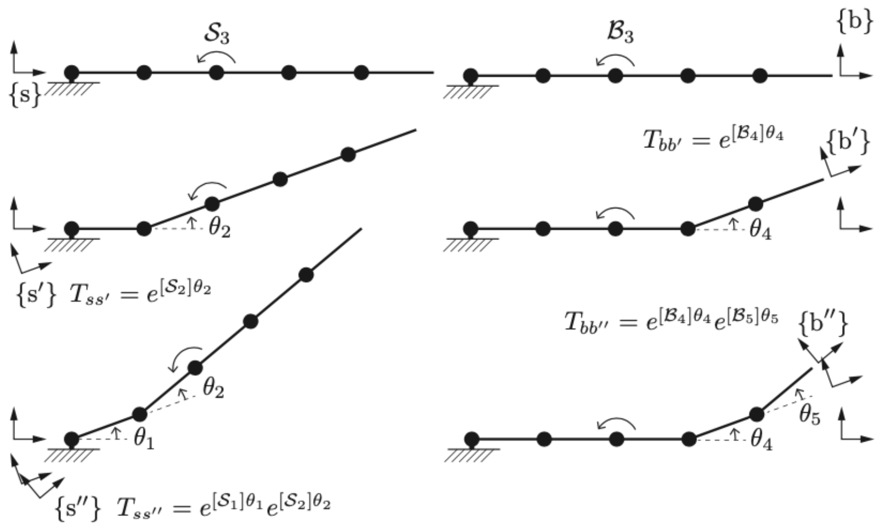

Figure 5.9: A 5R robot. (Left-hand column) Derivation of J_{s3}, the third column of the space Jacobian. (Right-hand column) Derivation of J_{b3}, the third column of the body Jacobian.

- Let us start with the third column, J_{s3}, of the space Jacobian. The joint variables θ_3, θ_4, and θ_5 have no impact on the spatial twist resulting from the joint velocity $\dot{\theta}_3$ b/c they do not displace axis 3 relative to {s}

$$T_{ss''} = e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}$$

- J_{s3} represents the screw relative to {s} for arbitrary joint angles θ_1 and θ_2 while $\mathcal{S}_3$ represents the screw relative to {s} at its zero position

$$J_{s3} = Ad_{T_{ss''}}(\mathcal{S}_3)$$

- Consider the third column, J_{b3}, of the body Jacobian. The joint variables θ_1, θ_2, and θ_3 have no impact on the spatial twist resulting from the joint velocity $\dot{\theta}_3$ b/c they do not displace axis 3 relative to {b}

$$T_{bb''} = e^{[\mathcal{B}_4]\theta_4} e^{[\mathcal{B}_5]\theta_5}$$

- Along with similar result, we have

$$\begin{aligned} J_{b3} &= Ad_{T_{b''b}}(\mathcal{B}_3) \\ &= Ad_{T^{-1}_{bb''}}(\mathcal{B}_3) \\ &= Ad_{e^{-[\mathcal{B}_5]\theta_5} e^{-[\mathcal{B}_4]\theta_4}}(\mathcal{B}_3) \end{aligned}$$

1.4 Relationship between the Space and Body Jacobian

- Consider the FK as $T_{sb}(\theta)$. The twist of the end-effector frame can be written in terms of the fixed- and end-effector-frame coordinates as

$$[\mathcal{V}_s] = \dot{T}_{sb}T_{sb}^{-1} \qquad\qquad [\mathcal{V}_b] = T_{sb}^{-1}\dot{T}_{sb}$$

  with the relation of $\mathcal{V}_s = Ad_{T_{sb}}(\mathcal{V}_b)$ and $\mathcal{V}_b = Ad_{T_{bs}}(\mathcal{V}_s)$.

- The twists are also related to their respective Jacobians via

$$\mathcal{V}_s = J_s(\theta)\dot{\theta} \qquad\qquad \mathcal{V}_b = J_b(\theta)\dot{\theta}$$

- By the following property

$$\begin{aligned}\mathcal{V}_b &= J_b(\theta)\dot{\theta}\\ &= Ad_{T_{bs}}(\mathcal{V}_s) = Ad_{T_{bs}}(J_s(\theta)\dot{\theta})\\ &= Ad_{T_{bs}}(J_s(\theta))\dot{\theta}\end{aligned}$$

  we have

$$J_b(\theta) = Ad_{T_{bs}}(J_s(\theta)) \qquad\qquad J_s(\theta) = Ad_{T_{sb}}(J_b(\theta))$$

- The fact that the space and body Jacobians, and the space and body twists, are similarly related by the adjoint map should not be surprising since each column of the space or body Jacobian corresponds to a twist.

- $J_b(\theta)$ and $J_s(\theta)$ always have the same rank.

1.5 Alternative Notions of the Jacobian

- When using a minimum set of coordinates, the end-effector velocity is not given by a twist $\mathcal{V}$ but by the time derivative of the coordinates $\dot{q}$, and the Jacobian J_a in the velocity kinematics

  $$\dot{q} = J_a(\theta)\dot{\theta}$$

  is sometimes called the analytic Jacobian as opposed to the geometric Jacobian in space and body form.

- For an $SE(3)$ task space, a typical choice of the minimal coordinates $q \in \Re^6$ includes three coordinates for the origin of the end-effector frame in the fixed frame and three coordinates for the orientation of the end-effector frame in the fixed frame.

- Example coordinates for the orientation include the Euler angles (see Appendix B) and exponential coordinates for rotation.

Example 5.3. *Find the analytic Jacobian J_a with exponential coordinates $r = \hat{\omega}\theta$ ($\|\hat{\omega}\| = 1$ and $\theta \in [0, \pi]$) for rotation from the body Jacobian J_b?*

- Consider an open chain with n joints and the body Jacobian. The angular and linear velocity components of $\mathcal{V}_b = (\omega_b, v_b)$ can be written explicitly as

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = J_b(\theta)\dot{\theta} = \begin{bmatrix} J_\omega(\theta) \\ J_v(\theta) \end{bmatrix} \dot{\theta}$$

- Suppose that our minimal set of coordinates $q \in \Re^6$ is given by $q = (r, x)$, where $x \in \Re^3$ is the position of the origin of the end-effector frame and $r = \hat{\omega}\theta \in \Re^3$ is the exponential coordinate representation for the rotation.

- The coordinate time derivative $\dot{x}$ is related to v_b by a rotation that gives v_b in the fixed coordinates:

$$\dot{x} = R_{sb}(\theta)v_b = R_{sb}(\theta)J_v(\theta)\dot{\theta}$$

  where $R_{sb}(\theta) = e^{[r]} = e^{[\hat{\omega}]\theta}$.

- The time-derivative $\dot{r}$ is related to the body angular velocity ω_b by

$$\omega_b = A(r)\dot{r} \qquad \rightarrow \qquad \dot{r} = A^{-1}(r)\omega_b = A^{-1}(r)J_\omega(\theta)\dot{\theta}$$

  provided that the matrix $A(r)$ is invertible, $\dot{r}$ can be obtained from ω_b where

$$A(r) = I - \frac{1 - \cos\|r\|}{\|r\|^2}[r] + \frac{\|r\| - \sin\|r\|}{\|r\|^3}[r]^2$$

- Putting these together, we have

$$\dot{q} = \begin{bmatrix} \dot{r} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} A^{-1}(r) & 0 \\ 0 & R_{sb}(\theta) \end{bmatrix} \begin{bmatrix} J_\omega(\theta) \\ J_v(\theta) \end{bmatrix} \dot{\theta}$$

- Analytic Jacobian is related to the body Jacobian

$$J_a(\theta) = \begin{bmatrix} A^{-1}(r) & 0 \\ 0 & R_{sb}(\theta) \end{bmatrix} J_b(\theta)$$

1.6 Looking Ahead to Inverse Velocity Kinematics

- The velocity FK can be written independently of the frame in which the twists are represented

$$\mathcal{V} = J(\theta)\dot{\theta}$$

- For given a desired twist $\mathcal{V}$, the velocity IK (inverse kinematics) is given by
  - if $J(\theta)$ is square ($n = 6$) and of full rank, then

$$\dot{\theta} = J^{-1}(\theta)\mathcal{V}$$

  - if $n < 6$, then arbitrary twists $\mathcal{V}$ cannot be achieved - the robot does not have enough joints.
  - if $n > 6$, then we call the robot redundant.

$$\dot{\theta} = J^{+}(\theta)\mathcal{V} + (I - J^{+}J)\eta$$

    where $J^{+} = J^{T}(JJ^{T})^{-1}$ so that $JJ^{+} = I$, and η is an arbitrary number to determine the rate of internal motion.
  - A desired twist $\mathcal{V}$ places six constraints on the joint rates, and the remaining $n-6$ freedoms correspond to internal motions of the robot that are not evident in the motion of the end-effector.