제 4 장

Forward Kinematics

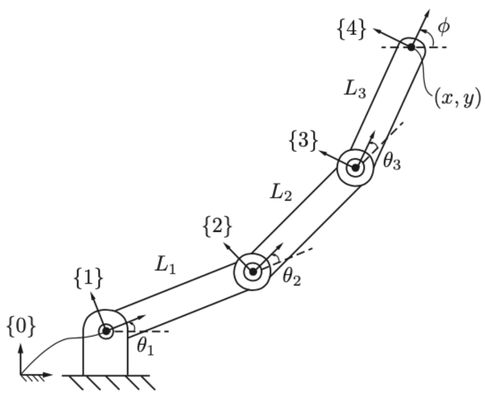

Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$- and $\hat{y}$-axis is shown; the $\hat{z}$-axes are parallel and out of the page.

The FK refers to the calculation of the position and orientation of its end-effector from its joint coordinates θ.
The Cartesian position (x, y) and orientation ϕ of the end-effector frame as functions of the joint angles $(\theta_1, \theta_2, \theta_3)$ are given by

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

$$\phi = \theta_1 + \theta_2 + \theta_3$$

1. After letting $\theta_1 = \theta_2 = \theta_3 = 0$, let us define M (it is called home position or zero position) to be the position and orientation of frame $\{4\}$

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. By letting $\theta_1 = \theta_2 = 0$, the screw axis corresponding to rotating about joint 3 can be expressed in the $\{0\}$ frame as

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} \qquad \text{where} \qquad \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

3. Let us take a visual inspection of Figure 4.1, when the arm is stretched out straight to the right at its zero configuration, imagine a turntable rotating with an angular velocity of $\omega_3 = 1$ rad/s about the axis of joint 3. The linear velocity v_3 of the point on the turntable at the origin of $\{0\}$ is in the $-\hat{y}_0$-direction at a rate of $L_1 + L_2$ units/s. Algebraically, $v_3 = \omega_3 \times (-q_3)$, where q_3 is any point on the axis of joint 3 expressed in $\{0\}$, e.g., $q_3 = (L_1 + L_2, 0, 0)$.

4. The screw axis $\mathcal{S}_3$ can be expressed in $se(3)$ matrix form as

$$[\mathcal{S}_3] = \begin{bmatrix} [\omega_3] & v_3 \\ 0_{3\times 1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & \omega_y & v_x \\ \omega_z & 0 & -\omega_x & v_y \\ -\omega_y & \omega_x & 0 & v_z \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. Let us take the matrix exponential representation for screw motion

$$e^{[\mathcal{S}_3]\theta_3} = \begin{bmatrix} e^{[\omega_3]\theta_3} & G(\theta_3)v_3 \\ 0_{3\times 1} & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & (L_1 + L_2)(1 - \cos\theta_3) \\ \sin\theta_3 & \cos\theta_3 & 0 & -(L_1 + L_2)\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & & 1 \end{bmatrix}$$

where

$$e^{[\omega_3]\theta_3} = I + \sin\theta_3[\omega_3] + (1 - \cos\theta_3)[\omega_3]^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta_3 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (1 - \cos\theta_3) \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G(\theta_3) = \theta_3 I + (1 - \cos\theta_3)[\omega_3] + (\theta_3 - \sin\theta_3)[\omega_3]^2$$

$$= \theta_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1 - \cos\theta_3) \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (\theta_3 - \sin\theta_3) \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sin\theta_3 & -(1 - \cos\theta_3) & 0 \\ (1 - \cos\theta_3) & \sin\theta_3 & 0 \\ 0 & 0 & \theta_3 \end{bmatrix}$$

6. Therefore, for any θ_3, the matrix exponential representation for screw motions from the previous chapter (pre-multiplication) allows us to write

$$\begin{aligned}
T_{04} &= e^{[\mathcal{S}_3]\theta_3} M \qquad \textbf{for } \theta_1 = \theta_2 = 0 \\
&= \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & (L_1+L_2)(1-\cos\theta_3) \\ \sin\theta_3 & \cos\theta_3 & 0 & -(L_1+L_2)\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L_1+L_2+L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_1+L_2+L_3\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & L_1+L_2+L_3\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & & 1 \end{bmatrix}
\end{aligned}$$

7. Similarly, for $\theta_1 = 0$ and any fixed (but arbitrary) θ_3, rotation about joint 2 can be viewed as applying a screw motion to the rigid (link 2)/(link 3) pair, i.e.,

$$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \qquad \textbf{for } \theta_1 = 0$$

the screw axis corresponding to rotating about joint 2 can be expressed in the $\{0\}$ frame as

$$\mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} \qquad \textbf{where} \qquad \omega_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ -L_1 \\ 0 \end{bmatrix}$$

8. Finally, keeping θ_2 and θ_3 fixed, rotation about joint 1 can be viewed as applying a screw motion to the entire rigid three-link assembly.

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

the screw axis corresponding to rotating about joint 1 can be expressed in the $\{0\}$ frame as

$$\mathcal{S}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} \qquad \text{where} \qquad \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

9. Several comments

   - The FK can be expressed as a product of matrix exponentials (PoE), each corresponding to a screw motion.
   - The PoE derivation of the forward kinematics does not use any link reference frames;
   - Another representation for the FK of open chains relies on the Denavit-Hartenberg parameters (D-H parameters).
   - The advantage of the D-H representation is that it requires the minimum number of parameters to describe the robot's kinematics: for an n-joint robot, it uses $3n$ numbers to describe the robot's structure and n numbers to describe the joint values.
   - The PoE representation is not minimal (it requires $6n$ numbers to describe the n screw axes, in addition to the n joint values), but it has advantages over the D-H representation (e.g., no link frames are necessary).

1 Product of Exponentials (PoE) Formula

To use the PoE formula,

- assign a stationary frame {s} (e.g., at the fixed base of the robot or anywhere else that is convenient for defining a reference frame)
- assign a frame {b} at the end-effector, described by M when the robot is at its zero position.
- refer to {s} as frame {0},
- use frames {i} for $i = 1, \cdots, n$ (the frames for links i at joints i),
- use one more frame {n + 1} (corresponding to {b}) at the end-effector.
- In some cases, frame {n + 1} simply refers to {n} as the end-effector frame {b}.

1.1 First Formulation: Screw Axes in the Base Frame (space form of PoE)

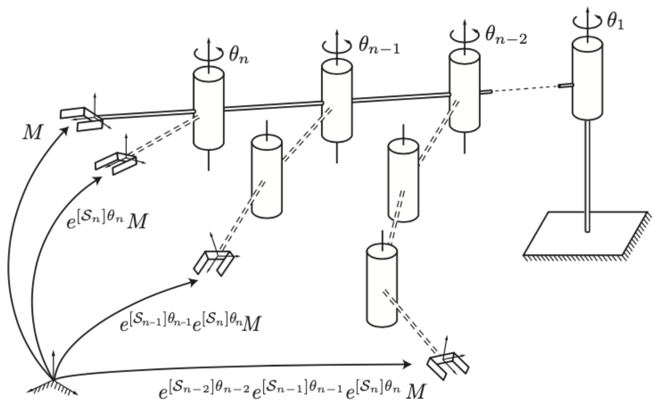

Figure 4.2: Illustration of the PoE formula for an n-link spatial open chain.

1. Consider a general spatial open chain consisting of n one-dof joints that are connected serially.
2. To solve the FK, you must choose a fixed base frame $\{s\}$ and an end-effector frame $\{b\}$ attached to the last link.
3. Place the robot in its zero position by setting all joint values to zero, with the direction of positive displacement (rotation for revolute joints, translation for prismatic joints) for each joint specified.
4. Let $M \in SE(3)$ denote the configuration of the end-effector frame relative to the fixed base frame when the robot is in its zero position.
5. Suppose that joint n is displaced to some joint value θ_n, then the end-effector frame M then undergoes a displacement of the form

$$T = e^{[\mathcal{S}_n]\theta_n} M$$

where $T \in SE(3)$ is the new configuration of the end-effector frame and $\mathcal{S}_n = (\omega_n, v_n)$ is the screw axis of joint n as expressed in the fixed base frame.

6. Continuing with this reasoning and now allowing all the joints $(\theta_1, \cdots, \theta_n)$ to vary, it follows that

$$T = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M$$

   This is the PoE formula describing the forward kinematics of an n-dof open chain. Specifically, it is called the space form of the PoE formula, referring to the fact that the screw axes are expressed in the fixed space frame.

7. To summarize, to calculate the forward kinematics of an open chain using the space form of the PoE formula, we need the following elements:

   a) the end-effector configuration $M \in SE(3)$ when the robot is at its home (or zero) position;

   b) the screw axes $\mathcal{S}_1, \cdots, \mathcal{S}_n$ expressed in the fixed base frame,

   c) the joint variables $\theta_1, \cdots, \theta_n$.

1.2 Examples

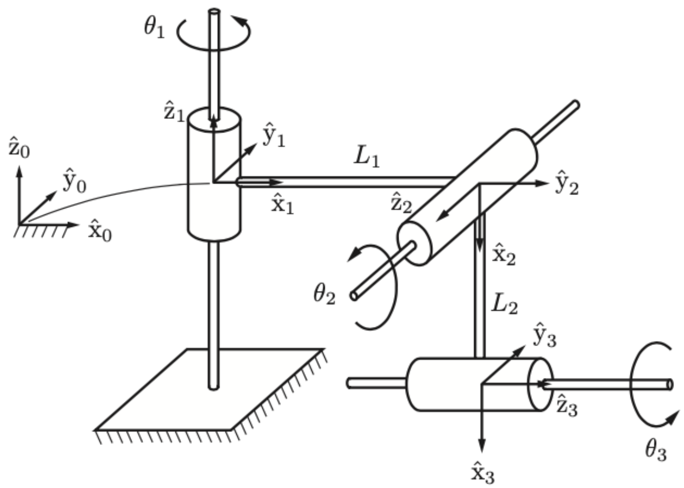

Figure 4.3: A 3R spatial open chain.

Example 4.1. *(3R spatial open chain). Consider the 3R open chain, shown in its home (or zero) position. Choose the fixed frame {0} and end-effector frame {3} as indicated in the figure, and express all vectors and homogeneous transformations in terms of the fixed frame.*

The forward kinematics using the space form of the PoE is defined with the end-effector frame configuration M at zero position and the screw axes:

$$T = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \quad \text{with} \quad M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(0,-1,0)	(0,0,-L_1)
3	(1,0,0)	(0,-L_2,0)

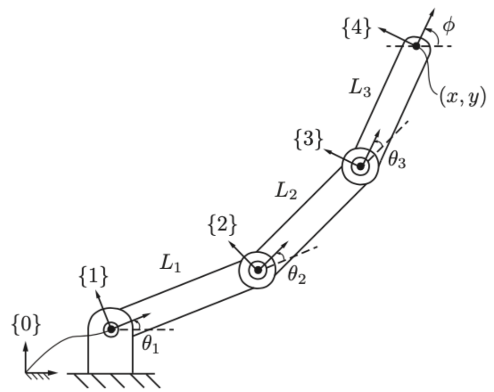

Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$- and $\hat{y}$-axis is shown; the $\hat{z}$-axes are parallel and out of the page.

Example 4.2. *(3R planar open chain). Find the FK in 3D and 2D space ?*

The forward kinematics in the 3D space has the form

$$T = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \quad \text{with} \quad M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(0,0,1)	(0,-L_1,0)
3	(0,0,1)	(0,$-(L_1 + L_2)$,0)

Since the motion is in the $\hat{x} - \hat{y}$ plane, we could equivalently write each screw axis $\mathcal{S}_i$ as a 3-vector (ω_z, v_x, v_y). The forward kinematics in the 2D space has

$$T = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \quad \text{with} \quad M = \begin{bmatrix} 1 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

i	ω_i	v_i
1	1	(0,0)
2	1	(0,-L_1)
3	1	(0,$-(L_1 + L_2)$)