# 2 Dynamics of a Single Rigid Body

## 2.1 Classical Formulation

- Consider a rigid body consisting of a number of rigidly connected point masses, where point mass $i$ has mass $\mathrm{m}_i$ and the total mass is $\mathrm{m} = \sum_i \mathrm{m}_i$.
- Let $r_i = (x_i, y_i, z_i)$ be the fixed location of mass $i$ in a body frame {b}, where the origin of this frame is the unique point such that

$$\sum_i \mathrm{m}_i r_i = 0$$

  This point is known as the center of mass (CoM). The body frame {b} should be chosen at CoM.

- Assume that the body is moving with a body twist $\mathcal{V}_b = (\omega_b, v_b)$, and let $p_i(t)$ be the time-varying position of $\mathrm{m}_i$, initially located at $r_i$, in the inertial frame {b}. Then

$$\begin{aligned}
\dot{p}_i &= v_b + \omega_b \times p_i \\
\ddot{p}_i &= \dot{v}_b + \dot{\omega}_b \times p_i + \omega_b \times \dot{p}_i = \dot{v}_b + \dot{\omega}_b \times p_i + \omega_b \times (v_b + \omega_b \times p_i) \\
&= \dot{v}_b + [\dot{\omega}_b] r_i + [\omega_b] v_b + [\omega_b]^2 r_i
\end{aligned}$$

  where $p_i$ is substituted with $r_i$

- Taking as a given that $f_i = \mathrm{m}_i \ddot{p}_i$ for a point mass, the force and moment acting on $\mathrm{m}_i$ are

$$\begin{aligned}
f_i &= \mathrm{m}_i (\dot{v}_b + [\dot{\omega}_b] r_i + [\omega_b] v_b + [\omega_b]^2 r_i) \\
m_i &= [r_i] f_i
\end{aligned}$$

- The total force and moment acting on the body is expressed as the wrench $\mathcal{F}_b$:

$$\mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \sum_i m_i \\ \sum_i f_i \end{bmatrix}$$

- To simplify the expressions for $f_b$ and $m_b$, keep in mind that $\sum_i m_i r_i = 0_{3\times 1}$ (and therefore $\sum_i m_i [r_i] = 0_{3\times 3}$) and, for $a, b \in \Re^3$, $[a] = -[a]^T$, $[a]b = -[b]a$, and $[a][b] = ([b][a])^T$.

- Focusing on the linear dynamics,

$$\begin{aligned} f_b &= \sum_i \mathrm{m}_i(\dot{v}_b + [\dot{\omega}_b]r_i + [\omega_b]v_b + [\omega_b]^2 r_i) \\ &= \sum_i \mathrm{m}_i(\dot{v}_b + [\omega_b]v_b) + \sum_i \mathrm{m}_i[\dot{\omega}_b]r_i + \sum_i \mathrm{m}_i[\omega_b][\omega_b]r_i \\ &= \sum_i \mathrm{m}_i(\dot{v}_b + [\omega_b]v_b) + [\dot{\omega}_b](\sum_i \mathrm{m}_i r_i) + [\omega_b][\omega_b](\sum_i \mathrm{m}_i r_i) \\ &= \mathrm{m}\dot{v}_b + [\omega_b]\mathrm{m}v_b \end{aligned}$$

- Now focusing on the rotational dynamics,

$$\begin{aligned}
m_b &= \sum_i [r_i] f_i \\
&= \sum_i \mathrm{m}_i [r_i](\dot{v}_b + [\dot{\omega}_b] r_i + [\omega_b] v_b + [\omega_b]^2 r_i) \\
&= (\sum_i \mathrm{m}_i [r_i])(\dot{v}_b + [\omega_b] v_b) + \sum_i \mathrm{m}_i [r_i][\dot{\omega}_b] r_i + \sum_i \mathrm{m}_i [r_i][\omega_b]^2 r_i \\
&= -\sum_i \mathrm{m}_i [r_i]^2 \dot{\omega}_b - \sum_i \mathrm{m}_i [r_i]^T [\omega_b]^T [r_i] \omega_b \\
&= -\sum_i \mathrm{m}_i [r_i]^2 \dot{\omega}_b - \sum_i \mathrm{m}_i [\omega_b][r_i]^2 \omega_b \\
&= \left(-\sum_i \mathrm{m}_i [r_i]^2\right) \dot{\omega}_b + [\omega_b] \left(-\sum_i \mathrm{m}_i [r_i]^2\right) \omega_b \\
&= \mathcal{I}_b \dot{\omega}_b + [\omega_b] \mathcal{I}_b \omega_b
\end{aligned}$$

  where $\mathcal{I}_b \in \Re^{3\times 3}$ is the body's rotational inertia matrix expressed in terms of the CoM. It is known as Euler's equation for a rotating rigid body.

- The rotational kinetic energy is given by the quadratic

$$\mathcal{K} = \frac{1}{2} \omega_b^T \mathcal{I}_b \omega_b$$

- One difference is that $\mathcal{I}_b$ is constant whereas the mass matrix $M(\theta)$ changes with the configuration of the mechanism.

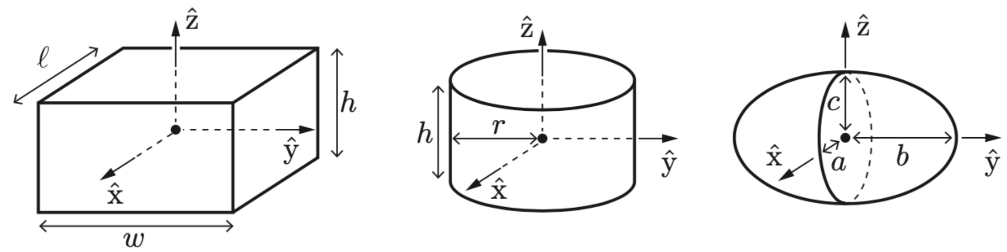
- Writing out the individual entries of $\mathcal{I}_b$, we get

$$\mathcal{I}_b = -\sum_i \mathrm{m}_i [r_i]^2 = -\sum_i \mathrm{m}_i \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_i \mathrm{m}_i(y_i^2 + z_i^2) & -\sum_i \mathrm{m}_i x_i y_i & -\sum_i \mathrm{m}_i x_i z_i \\ -\sum_i \mathrm{m}_i x_i y_i & \sum_i \mathrm{m}_i(x_i^2 + z_i^2) & -\sum_i \mathrm{m}_i y_i z_i \\ -\sum_i \mathrm{m}_i x_i z_i & -\sum_i \mathrm{m}_i y_i z_i & \sum_i \mathrm{m}_i(x_i^2 + y_i^2) \end{bmatrix} = \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix}$$

- The summations can be replaced by volume integrals over the body $\mathcal{B}$, using the differential volume element $dV$ , with the point masses $\mathrm{m}_i$ replaced by a mass density function $\rho(x, y, z) = \frac{\mathrm{m}}{V}$:

$$\mathcal{I}_{xx} = \int_{\mathcal{B}} (y^2 + z^2)\rho(x, y, z)dV$$
$$\mathcal{I}_{yy} = \int_{\mathcal{B}} (x^2 + z^2)\rho(x, y, z)dV$$
$$\mathcal{I}_{zz} = \int_{\mathcal{B}} (x^2 + y^2)\rho(x, y, z)dV$$
$$\mathcal{I}_{xy} = -\int_{\mathcal{B}} xy\rho(x, y, z)dV$$
$$\mathcal{I}_{xz} = -\int_{\mathcal{B}} xz\rho(x, y, z)dV$$
$$\mathcal{I}_{yz} = -\int_{\mathcal{B}} yz\rho(x, y, z)dV$$




| rectangular parallelepiped: | circular cylinder: | ellipsoid: |
| --- | --- | --- |
| volume $= abc$, | volume $= \pi r^2 h$, | volume $= 4\pi abc/3$, |
| $\mathcal{I}_{xx} = \mathfrak{m}(w^2 + h^2)/12$, | $\mathcal{I}_{xx} = \mathfrak{m}(3r^2 + h^2)/12$, | $\mathcal{I}_{xx} = \mathfrak{m}(b^2 + c^2)/5$, |
| $\mathcal{I}_{yy} = \mathfrak{m}(\ell^2 + h^2)/12$, | $\mathcal{I}_{yy} = \mathfrak{m}(3r^2 + h^2)/12$, | $\mathcal{I}_{yy} = \mathfrak{m}(a^2 + c^2)/5$, |
| $\mathcal{I}_{zz} = \mathfrak{m}(\ell^2 + w^2)/12$ | $\mathcal{I}_{zz} = \mathfrak{m}r^2/2$ | $\mathcal{I}_{zz} = \mathfrak{m}(a^2 + b^2)/5$ |

**Figure 8.5:** The principal axes and the inertia about the principal axes for uniform-density bodies of mass $\mathfrak{m}$. Note that the $\hat{x}$ and $\hat{y}$ principal axes of the cylinder are not unique.

- If the body has uniform density, $\mathcal{I}_b$ is determined exclusively by the shape of the rigid body
- One principal axis maximizes the moment of inertia among all axes passing through the CoM, and another minimizes the moment of inertia.
- If the principal axes of inertia are aligned with the axes of {b}, the off-diagonal terms of $\mathcal{I}_b$ are all zero, and the eigenvalues are the scalar moments of inertia $\mathcal{I}_{xx}$, $\mathcal{I}_{yy}$ and $\mathcal{I}_{zz}$ about the $x$-, $y$-, and $z$-axes, respectively. In this case, the equations of motion simplify to

$$\begin{aligned} m_b &= \mathcal{I}_b\dot{\omega}_b + [\omega_b]\mathcal{I}_b\omega_b \\ &= \begin{bmatrix} \mathcal{I}_{xx}\dot{\omega}_x + (\mathcal{I}_{zz} - \mathcal{I}_{yy})\omega_y\omega_z \\ \mathcal{I}_{yy}\dot{\omega}_y + (\mathcal{I}_{xx} - \mathcal{I}_{zz})\omega_x\omega_z \\ \mathcal{I}_{zz}\dot{\omega}_z + (\mathcal{I}_{yy} - \mathcal{I}_{xx})\omega_x\omega_y \end{bmatrix} \end{aligned}$$

  where $\omega_b = (\omega_x, \omega_y, \omega_z)$.

- Inertia matrix $\mathcal{I}_b$ can be expressed in a rotated frame {c} described by the rotation matrix $R_{bc}$.
- Since the kinetic energy of the rotating body is independent of the chosen frame, we have

$$\frac{1}{2}\omega_c^T\mathcal{I}_c\omega_c = \frac{1}{2}\omega_b^T\mathcal{I}_b\omega_b = \frac{1}{2}(R_{bc}\omega_c)^T\mathcal{I}_b(R_{bc}\omega_c) = \frac{1}{2}\omega_c^T(R_{bc}^T\mathcal{I}_bR_{bc})\omega_c$$

- In other words,

$$\mathcal{I}_c = R_{bc}^T\mathcal{I}_bR_{bc}$$

  If the axes of {b} are not aligned with the principal axes of inertia then we can diagonalize the inertia matrix by expressing it instead in the rotated frame {c}, where the columns of $R_{bc}$ correspond to the eigenvalues of $\mathcal{I}_b$.

- Sometimes it is convenient to represent the inertia matrix in a frame at a point not at the center of mass of the body, for example at a joint.

  **Theorem 8.1.** *(Steiner's theorem) The inertia matrix $\mathcal{I}_q$ about a frame aligned with $\{b\}$, but at a point $q = (q_x, q_y, q_z)$ in $\{b\}$, is related to the inertia matrix $\mathcal{I}_b$ calculated at the CoM by*

  $$\mathcal{I}_q = \mathcal{I}_b + \mathrm{m}(q^T q I_{3\times 3} - qq^T)$$

  *where $I$ is the $3 \times 3$ identity matrix and $\mathrm{m}$ is the mass of the body.*

- Steiner's theorem is a more general statement of the parallel-axis theorem, which states that the scalar inertia $\mathcal{I}_d$ about an axis parallel to, but a distance $d$ from, an axis through the CoM is related to the scalar inertia $\mathcal{I}_{cm}$ about the axis through the CoM by

  $$\mathcal{I}_d = \mathcal{I}_{cm} + \mathrm{m}d^2$$

- In the case of motion confined to the $x$-$y$-plane, where $\omega_b = (0, 0, \omega_z)$ and the inertia of the body about the $z$-axis through the CoM is given by the scalar $\mathcal{I}_{zz}$, the spatial rotational dynamics reduces to the planar rotational dynamics

  $$m_z = \mathcal{I}_{zz}\dot{\omega}_z$$

  and the rotational kinetic energy is

  $$\mathcal{K} = \frac{1}{2}\mathcal{I}_{zz}\omega_z^2$$

## 2.2 Twist-Wrench Formulation

- The linear dynamics $f_b = \mathrm{m}\dot{v}_b + [\omega_b]\mathrm{m}v_b$ and the rotational dynamics $m_b = \mathcal{I}_b\dot{\omega}_b + [\omega_b]\mathcal{I}_b\omega_b$ can be written in the following combined form:

$$\begin{bmatrix} m_b \\ f_b \end{bmatrix} = \begin{bmatrix} \mathcal{I}_b & 0_{3\times3} \\ 0_{3\times3} & \mathrm{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} + \begin{bmatrix} [\omega_b] & 0_{3\times3} \\ 0_{3\times3} & [\omega_b] \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0_{3\times3} \\ 0_{3\times3} & \mathrm{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}$$

  where $I$ is the $3 \times 3$ identity matrix.

- With the benefit of hindsight, and also making use of the fact that $[v]v = v \times v = 0$ and $[v]^T = -[v]$, we can write Equation in the following equivalent form:

$$\begin{aligned} \begin{bmatrix} m_b \\ f_b \end{bmatrix} &= \begin{bmatrix} \mathcal{I}_b & 0_{3\times3} \\ 0_{3\times3} & \mathrm{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} + \begin{bmatrix} [\omega_b] & [v_b] \\ 0_{3\times3} & [\omega_b] \end{bmatrix} \begin{bmatrix} \mathcal{I}_b & 0_{3\times3} \\ 0_{3\times3} & \mathrm{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{I}_b & 0_{3\times3} \\ 0_{3\times3} & \mathrm{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} - \begin{bmatrix} [\omega_b] & 0_{3\times3} \\ [v_b] & [\omega_b] \end{bmatrix}^T \begin{bmatrix} \mathcal{I}_b & 0_{3\times3} \\ 0_{3\times3} & \mathrm{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \end{aligned}$$

- Written this way, each term can now be identified with six-dimensional spatial quantities as follows:

  1. The vectors $(\omega_b, v_b)$ and $(m_b, f_b)$ can be respectively identified with the body twist $\mathcal{V}_b$ and body wrench $\mathcal{F}_b$,

  $$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \qquad \mathcal{F}_b = \begin{bmatrix} m_b \\ f_b \end{bmatrix}$$

  2. The spatial inertia matrix $\mathcal{G}_b \in \Re^{6\times 6}$ is defined as

  $$\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0_{3\times 3} \\ 0_{3\times 3} & \mathrm{m}I \end{bmatrix}$$

  As an aside, the kinetic energy of the rigid body can be expressed in terms of the spatial inertia matrix as

  $$\text{kinetic energy} = \frac{1}{2}\omega_b^T \mathcal{I}_b \omega_b + \frac{1}{2}\mathrm{m}v_b^T v_b = \frac{1}{2}\mathcal{V}_b^T \mathcal{G}_b \mathcal{V}_b$$

  3. The spatial momentum $\mathcal{P}_b \in \Re^6$ is defined as

  $$\mathcal{P}_b = \mathcal{G}_b \mathcal{V}_b$$

- The following matrix can be thought of as a generalization of the cross-product operation to six-dimensional twists.

$$-\begin{bmatrix}[\omega_b] & 0_{3\times3}\\ [v_b] & [\omega_b]\end{bmatrix}^T$$

First, recall that the cross product of two vectors $\omega_1, \omega_2 \in \Re^3$ can be calculated, using the skew-symmetric matrix notation, as follows:

$$[\omega_1 \times \omega_2] = [[\omega_1]\omega_2] = [\omega_1][\omega_2] - [\omega_2][\omega_1]$$

Second, given two twists $\mathcal{V}_1 = (\omega_1, v_1)$ and $\mathcal{V}_2 = (\omega_2, v_2)$, we perform a calculation analogous to the above:

$$\begin{aligned}[\mathcal{V}_1][\mathcal{V}_2] - [\mathcal{V}_2][\mathcal{V}_1] &= \begin{bmatrix}[\omega_1] & v_1\\ 0_{1\times3} & 0\end{bmatrix}\begin{bmatrix}[\omega_2] & v_2\\ 0_{1\times3} & 0\end{bmatrix} - \begin{bmatrix}[\omega_2] & v_2\\ 0_{1\times3} & 0\end{bmatrix}\begin{bmatrix}[\omega_1] & v_1\\ 0_{1\times3} & 0\end{bmatrix}\\ &= \begin{bmatrix}[\omega_1][\omega_2] - [\omega_2][\omega_1] & [\omega_1]v_2 - [\omega_2]v_1\\ 0_{1\times3} & 0\end{bmatrix} = \begin{bmatrix}[\omega'] & v'\\ 0_{1\times3} & 0\end{bmatrix}\end{aligned}$$

which can be written more compactly in vector form as

$$\begin{bmatrix}\omega'\\ v'\end{bmatrix} = \begin{bmatrix}[\omega_1] & 0_{3\times3}\\ [v_1] & [\omega_1]\end{bmatrix}\begin{bmatrix}\omega_2\\ v_2\end{bmatrix}$$

Generalization of the cross product to two twists $\mathcal{V}_1$ and $\mathcal{V}_2$ is called the Lie bracket of $\mathcal{V}_1$ and $\mathcal{V}_2$.

**Definition 8.1.** *Given two twists $\mathcal{V}_1 = (\omega_1, v_1)$ and $\mathcal{V}_2 = (\omega_2, v_2)$, the Lie bracket of $\mathcal{V}_1 = (\omega_1, v_1)$ and $\mathcal{V}_2 = (\omega_2, v_2)$, written either as $[ad_{\mathcal{V}_1}]\mathcal{V}_2$ or $ad_{\mathcal{V}_1}(\mathcal{V}_2)$, is defined as follows:*

$$\begin{bmatrix} [\omega_1] & 0_{3\times 3} \\ [v_1] & [\omega_1] \end{bmatrix} \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = [ad_{\mathcal{V}_1}]\mathcal{V}_2 = ad_{\mathcal{V}_1}(\mathcal{V}_2) \in \Re^6$$

*where*

$$[ad_{\mathcal{V}}] = \begin{bmatrix} [\omega] & 0_{3\times 3} \\ [v] & [\omega] \end{bmatrix} \in \Re^{6\times 6}$$

- Using the notation and definitions above, the dynamic equations for a single rigid body can now be written as

$$\begin{aligned} \begin{bmatrix} m_b \\ f_b \end{bmatrix} &= \begin{bmatrix} \mathcal{I}_b & 0_{3\times 3} \\ 0_{3\times 3} & \mathrm{m}I \end{bmatrix} \begin{bmatrix} \dot{\omega}_b \\ \dot{v}_b \end{bmatrix} - \begin{bmatrix} [\omega_b] & 0_{3\times 3} \\ [v_b] & [\omega_b] \end{bmatrix}^T \begin{bmatrix} \mathcal{I}_b & 0_{3\times 3} \\ 0_{3\times 3} & \mathrm{m}I \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \\ \mathcal{F}_b &= \mathcal{G}_b\dot{\mathcal{V}}_b - ad^T_{\mathcal{V}_b}(\mathcal{P}_b) \\ &= \mathcal{G}_b\dot{\mathcal{V}}_b - [ad_{\mathcal{V}_b}]^T\mathcal{G}_b\mathcal{V}_b \end{aligned}$$

- Note the analogy between above Equation and the moment equation for a rotating rigid body:

$$\begin{aligned} m_b &= \mathcal{I}_b\dot{\omega}_b + [\omega_b]\mathcal{I}_b\omega_b \\ &= \mathcal{I}_b\dot{\omega}_b - [\omega_b]^T\mathcal{I}_b\omega_b \end{aligned}$$

**Definition 8.2.** *Given a twist $\mathcal{V} = (\omega, v)$ and a wrench $\mathcal{F} = (m, f)$, define the mapping*

$$ad_{\mathcal{V}}^T(\mathcal{F}) = [ad_{\mathcal{V}}]^T\mathcal{F} = \begin{bmatrix} [\omega] & 0_{3\times 3} \\ [v] & [\omega] \end{bmatrix}^T \begin{bmatrix} m \\ f \end{bmatrix} = \begin{bmatrix} -[\omega]m - [v]f \\ -[\omega]f \end{bmatrix}$$

## 2.3 Dynamics in Other Frames

- The derivation of the dynamic equations relies on the use of a CoM frame {b}.
- It is straightforward to express the dynamics in other frames. Let's call one such frame {a}.
- Since the kinetic energy of the rigid body must be independent of the frame of representation,

$$\begin{aligned}
\frac{1}{2}\mathcal{V}_a^T\mathcal{G}_a\mathcal{V}_a &= \frac{1}{2}\mathcal{V}_b^T\mathcal{G}_b\mathcal{V}_b \\
&= \frac{1}{2}([Ad_{T_{ba}}]\mathcal{V}_a)^T\mathcal{G}_b([Ad_{T_{ba}}]\mathcal{V}_a) \\
&= \frac{1}{2}\mathcal{V}_a^T([Ad_{T_{ba}}]^T\mathcal{G}_b[Ad_{T_{ba}}])\mathcal{V}_a
\end{aligned}$$

for the adjoint representation $Ad_T$ using $T = \begin{bmatrix} R & p \\ 0_{3\times 1} & 1 \end{bmatrix}$

$$Ad_T = \begin{bmatrix} R & 0_{3\times 3} \\ [p]R & R \end{bmatrix}$$

- In other words, the spatial inertia matrix $\mathcal{G}_a$ in {a} is related to $\mathcal{G}_b$ by

$$\mathcal{G}_a = [Ad_{T_{ba}}]^T \mathcal{G}_b [Ad_{T_{ba}}]$$

  This is a generalization of Steiner's theorem.

- Using the spatial inertia matrix $\mathcal{G}_a$, the equations of motion in the {b} frame can be expressed equivalently in the {a} frame as

$$\mathcal{F}_a = \mathcal{G}_a \dot{\mathcal{V}}_a - [ad_{\mathcal{V}_a}]^T \mathcal{G}_a \mathcal{V}_a$$

  The equations of motion is independent of the frame of representation.