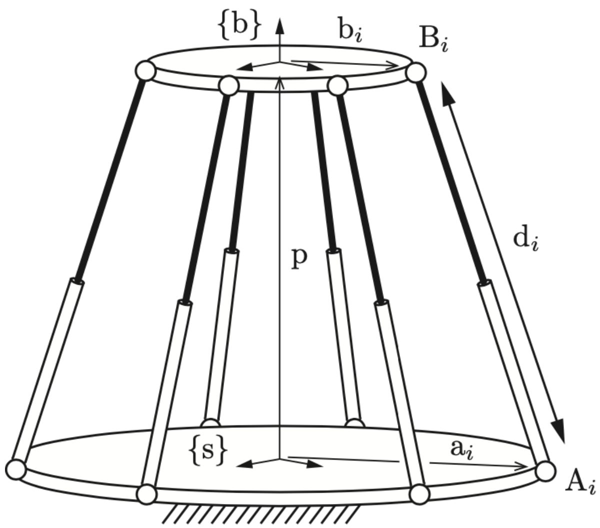
2 Differential Kinematics

- Unlike the case for open chains, in which the objective is to relate the input joint velocities to the twist of the end-effector frame, the analysis for closed chains is complicated by the fact that not all the joints are actuated.
- Only the actuated joints can be prescribed input velocities; the velocities of the remaining passive joints must then be determined from the kinematic constraint equations.
- These passive joint velocities are usually required in order to eventually determine the twist of the closed chain's end-effector frame.
- For open chains, the FK Jacobian is central to the velocity and static analysis.
- For closed chains, in addition to the FK Jacobian, the constraint Jacobian defined by the kinematic constraint equations - also plays a central role in the velocity and static analysis.
- Usually there are features of the mechanism that can be exploited to simplify and reduce the procedure for obtaining the two Jacobians.
- For the Stewart-Gough platform, the IK Jacobian can be obtained straightforwardly via static analysis.



(a) Stewart–Gough platform.

2.1 Stewart-Gough Platform

- In the previous slide, the IK for the Stewart-Gough platform can be solved analytically.
- Given the body-frame orientation $R \in SO(3)$ and position $p \in \Re^3$, the leg lengths $s \in \Re^6$ can be obtained analytically in the functional form

$$s = g(R, p) \quad \leftarrow \quad s_i^2 = d_i^T d_i = (p + Rb_i - a_i)^T (p + Rb_i - a_i) \text{ for } i = 1, 2, \cdots, 6$$

- In principle one could differentiate this equation and manipulate it into the form

$$\dot{s} = G(R, p)\mathcal{V}_s$$

where $\dot{s} \in \Re^6$ denotes the leg velocities, $\mathcal{V}_s \in \Re^6$ is the spatial twist, and $G(R, p) \in \Re^{6\times 6}$ is the Jacobian of the IK. (It will require considerable algebraic manipulation)

- As a different approach, the conservation of power principle is used to determine the static relationship $\tau = J^T\mathcal{F}$ for open chains.

- In the absence of external forces, the only forces applied to the moving platform occur at the spherical joints in the SPS mechanism.

- Let f_i be the three-dimensional linear force applied by leg i

$$f_i = \hat{n}_i\tau_i$$

where $\hat{n}_i \in \Re^3$ is a unit vector indicating the direction of the applied force and $\tau_i \in \Re$ is the magnitude of the linear force.

- The moment m_i generated by f_i is

$$m_i = r_i \times f_i = a_i \times f_i$$

where $r_i \in \Re^3$ denotes the vector from the {s}-frame origin to the point of application of the force (the location of spherical joint i in this case, so it can be replaced as a constant vector a_i at the fixed platform)

- Since neither the spherical joint at the moving platform nor the spherical joint at the fixed platform can resist any torques about them, the force f_i must be along the line of the leg.

- Combining f_i and m_i into the six-dimensional wrench $\mathcal{F}_i = (m_i, f_i)$, the resultant wrench $\mathcal{F}_s$ on the moving platform is given by

$$\begin{aligned}
\mathcal{F}_s &= \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4 + \mathcal{F}_5 + \mathcal{F}_6 \\
&= \begin{bmatrix} a_1 \times f_1 \\ f_1 \end{bmatrix} + \begin{bmatrix} a_2 \times f_2 \\ f_2 \end{bmatrix} + \begin{bmatrix} a_3 \times f_3 \\ f_3 \end{bmatrix} + \begin{bmatrix} a_4 \times f_4 \\ f_4 \end{bmatrix} + \begin{bmatrix} a_5 \times f_5 \\ f_5 \end{bmatrix} + \begin{bmatrix} a_6 \times f_6 \\ f_6 \end{bmatrix} \\
&= \begin{bmatrix} a_1 \times \hat{n}_1 \tau_1 \\ \hat{n}_1 \tau_1 \end{bmatrix} + \begin{bmatrix} a_2 \times \hat{n}_2 \tau_2 \\ \hat{n}_2 \tau_2 \end{bmatrix} + \begin{bmatrix} a_3 \times \hat{n}_3 \tau_3 \\ \hat{n}_3 \tau_3 \end{bmatrix} + \begin{bmatrix} a_4 \times \hat{n}_4 \tau_4 \\ \hat{n}_4 \tau_4 \end{bmatrix} + \begin{bmatrix} a_5 \times \hat{n}_5 \tau_5 \\ \hat{n}_5 \tau_5 \end{bmatrix} + \begin{bmatrix} a_6 \times \hat{n}_6 \tau_6 \\ \hat{n}_6 \tau_6 \end{bmatrix} \\
&= \begin{bmatrix} a_1 \times \hat{n}_1 \\ \hat{n}_1 \end{bmatrix} \tau_1 + \begin{bmatrix} a_2 \times \hat{n}_2 \\ \hat{n}_2 \end{bmatrix} \tau_2 + \begin{bmatrix} a_3 \times \hat{n}_3 \\ \hat{n}_3 \end{bmatrix} \tau_3 + \begin{bmatrix} a_4 \times \hat{n}_4 4 \\ \hat{n}_4 \end{bmatrix} \tau_+ \begin{bmatrix} a_5 \times \hat{n}_5 \\ \hat{n}_5 \end{bmatrix} \tau_5 + \begin{bmatrix} a_6 \times \hat{n}_6 \\ \hat{n}_6 \end{bmatrix} \tau_6 \\
&= \begin{bmatrix} a_1 \times \hat{n}_1 & \cdots & a_6 \times \hat{n}_6 \\ \hat{n}_1 & \cdots & \hat{n}_6 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_6 \end{bmatrix} \\
&= J_s^{-T} \tau
\end{aligned}$$

where J_s is the spatial Jacobian of the FK, and its inverse is given by

$$J_s^{-1} = \begin{bmatrix} a_1 \times \hat{n}_1 & \cdots & a_6 \times \hat{n}_6 \\ \hat{n}_1 & \cdots & \hat{n}_6 \end{bmatrix}^T = \begin{bmatrix} \hat{n}_1 \times (-a_1) & \cdots & \hat{n}_6 \times (-a_6) \\ \hat{n}_1 & \cdots & \hat{n}_6 \end{bmatrix}^T$$

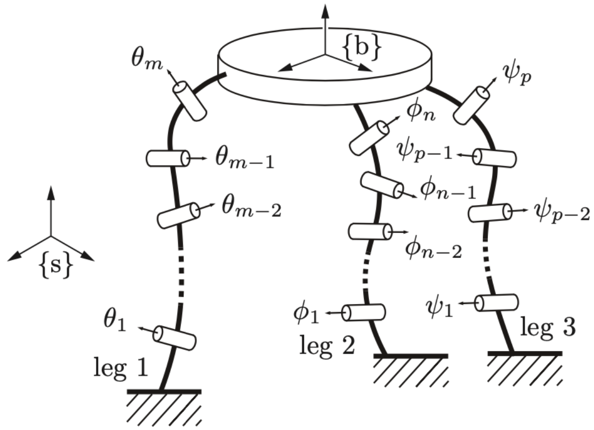


Figure 7.4: A general parallel mechanism.

2.2 General Parallel Mechanisms

- For the Stewart-Gough platform, the inverse Jacobian can be derived in terms of the screws associated with each straight-line leg.
- Consider more general parallel mechanisms where the static analysis is less straightforward.
- A procedure for determining the FK Jacobian that can be generalized to other types of parallel mechanisms.
- For simplicity, we will take $m = n = p = 5$ in the general three-leg mechanism, so that the mechanism has $dof = n + m + p - 12 = 3$.

- For the fixed and body frames indicated in the figure, we can write the FK for the three chains

$$T_1(\theta_1, \cdots, \theta_5) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_5]\theta_5} M_1$$
$$T_2(\phi_1, \cdots, \phi_5) = e^{[\mathcal{P}_1]\phi_1} \cdots e^{[\mathcal{P}_5]\phi_5} M_2$$
$$T_3(\psi_1, \cdots, \psi_5) = e^{[\mathcal{Q}_1]\psi_1} \cdots e^{[\mathcal{Q}_5]\psi_5} M_3$$

- The kinematic loop constraints can be expressed as

$$T_1(\theta) = T_2(\phi) \qquad\qquad T_2(\phi) = T_3(\psi)$$

- Since these constraints must be satisfied at all times, we can express their time derivatives in terms of their spatial twists, using

$$\dot{T}_1 T_1^{-1}(\theta) = \dot{T}_2 T_2^{-1}(\phi) \qquad\qquad \dot{T}_2 T_2^{-1}(\phi) = \dot{T}_3 T_3^{-1}(\psi)$$

- Since $\dot{T}_i T_i^{-1} = [\mathcal{V}_i]$, where $\mathcal{V}_i$ is the spatial twist of chain i's end-effector frame, the above identities can also be expressed in terms of the FK Jacobian for each chain:

$$J_1(\theta)\dot{\theta} = J_2(\phi)\dot{\phi} \qquad\qquad J_2(\phi)\dot{\phi} = J_3(\psi)\dot{\psi}$$

which can be rearranged as

$$\begin{bmatrix} J_1(\theta) & -J_2(\phi) & 0 \\ 0 & -J_2(\phi) & J_3(\psi) \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = 0$$

- Now we rearrange the 15 joints into those that are actuated and those that are passive. Assume without loss of generality that the three actuated joints are $(\dot{\theta}_1, \dot{\phi}_1, \dot{\psi}_1)$. Define the vector of the actuated joints $q_a \in \Re^3$ and the vector of the passive joints $q_p \in \Re^{12}$ as

$$q_a = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi}_1 \\ \dot{\psi}_1 \end{bmatrix} \qquad q_p = \begin{bmatrix} \dot{\theta}_2 \\ \vdots \\ \dot{\psi}_5 \end{bmatrix} \in \Re^{12} \qquad q = \begin{bmatrix} q_a \\ q_p \end{bmatrix} \in \Re^{15}$$

- Above equation can now be rearranged into the form

$$\begin{bmatrix} H_a(q) & H_p(q) \end{bmatrix} \begin{bmatrix} \dot{q}_a \\ \dot{q}_p \end{bmatrix} = 0 \quad \rightarrow \quad H_a \dot{q}_a + H_p \dot{q}_p = 0 \quad \rightarrow \quad \dot{q}_p = -H_p^{-1} H_a \dot{q}_a$$

- Assuming that H_p is invertible, once the velocities of the actuated joints are given, then the velocities of the remaining passive joints can be obtained uniquely

- It still remains to derive the FK Jacobian with respect to the actuated joints, i.e., to find $J_a(q) \in \Re^{6\times 3}$ satisfying $\mathcal{V}_s = J_a(q)\dot{q}_a$, where $\mathcal{V}_s$ is the spatial twist of the end-effector frame.

- For this purpose we can use the FK for any of the three open chains: for example, in terms of chain 1, $J_1(\theta)\dot{\theta} = \mathcal{V}_s$, and from $\dot{q}_p = -H_p^{-1} H_a \dot{q}_a$ we can write

$$\dot{\theta}_2 = g_2^T \dot{q}_a \qquad \dot{\theta}_3 = g_3^T \dot{q}_a \qquad \dot{\theta}_4 = g_4^T \dot{q}_a \qquad \dot{\theta}_5 = g_5^T \dot{q}_a$$

  where each $g_i(q) \in \Re^3$, for $i = 2, \cdots, 5$, can be obtained from $\dot{q}_p = -H_p^{-1} H_a \dot{q}_a$

- Defining the row vector $e_1^T = [100]$, the differential FK for chain 1 can now be written

$$\mathcal{V}_s = J_1(\theta) \begin{bmatrix} e_1^T \\ g_2^T \\ g_3^T \\ g_4^T \\ g_5^T \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\phi}_1 \\ \dot{\psi}_1 \end{bmatrix}$$

- Since we are seeking $J_a(q)$ in $\mathcal{V}_s = J_a(q)\dot{q}_a$, and since $\dot{q}_a = (\dot{\theta}_1, \dot{\phi}_1, \dot{\psi}_1)$, from the above it now follows that

$$J_a(q) = J_1(\theta) \begin{bmatrix} e_1^T \\ g_2^T \\ g_3^T \\ g_4^T \\ g_5^T \end{bmatrix}$$

  this equation could also have been derived using either chain 2 or chain 3.

- Given values for the actuated joints q_a, we still need to solve for the passive joints q_p from the loop-constraint equations. $\dot{q}_p = -H_p^{-1} H_a \dot{q}_a$
- The second point to note is that $H_p(q)$ may become singular, in which case $\dot{q}_p$ cannot be obtained from $\dot{q}_a$.
- Configurations in which $H_p(q)$ becomes singular correspond to actuator singularities.