# (PID) 5 $\mathcal{H}_\infty$ Optimality of PID Control

1. $\mathcal{H}_\infty$ (or $\mathcal{L}_2$-gain) optimality of a PID controller is analyzed, especially, for Lagrangian systems.

2. $\mathcal{H}_\infty$ norm ($\mathcal{L}_2$-gain) optimization is a means to reduce the disturbance effect in achieving the optimal performance cost.

3. For the $\mathcal{H}_\infty$ norm optimization from the disturbance effect to the performance cost, the nonlinear $\mathcal{H}_\infty$ control theory is utilized.

4. Nonlinear $\mathcal{H}_\infty$ control scheme

   - is robust and performs well
   - has not been widely accepted in industry
   - requires the solution of nonlinear partial differential equation (PDE) (or HJI equation)
   - full state feedback case [A. J. van der Schaft : 1992]
   - output feedback case [A. Isidori et al : 1992]

# (PID) 5.1 Error State-Space Representation of Lagrangian System

1. (Lagrangian system) It is described with $n$ generalized coordinates $q = [q_1, q_2, \cdots, q_n]^T \in \Re^n$:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) + d(t) = \tau, \tag{109}$$

   where

   - $M(q) \in \Re^{n \times n}$ is an inertia matrix,
   - $C(q, \dot{q}) \in \Re^{n \times n}$ Coriolis and centrifugal matrix,
   - $g(q) \in \Re^n$ gravitational torque vector,
   - $\tau \in \Re^n$ control input torque vector
   - $d(t)$ unknown external disturbance vector.

2. (Definition of Extended Disturbance) As a new form, let us define the *extended disturbance* for the trajectory tracking control, including the external disturbance, as follows form:

$$w\left(t, \dot{e}, e, \int edt\right) = M(q)\left(\ddot{q}_d + K_P\dot{e} + K_I e\right) + C(q, \dot{q})\left(\dot{q}_d + K_P e + K_I \int edt\right) + g(q) + d(t), \tag{110}$$

   where

   - $K_P$ and $K_I$ are diagonal constant matrices,
   - $e = q_d - q$ is the configuration error.

3. (Trajectory Tracking System Model) If the extended disturbance defined above is used in the Lagrangian system of Eq. (109), then the trajectory tracking system model can be rewritten as

$$M(q)\dot{s} + C(q,\dot{q})s = w\left(t, \dot{e}, e, \int edt\right) + u, \tag{111}$$

where $u \triangleq -\tau$ and $s \triangleq \dot{e} + K_P e + K_I \int edt$ .

4. (Error State Space Representation of Trajectory Tracking System Model) If the state vector is defined for the tracking system model (111) as follows:

$$x \triangleq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \int e dt \\ e \\ \dot{e} \end{bmatrix} \in \Re^{3n} \tag{112}$$

then the state space representation of trajectory tracking system model can be obtained as the following form:

$$\dot{x} = A(x,t)x + B(x,t)w + B(x,t)u, \tag{113}$$

where

$$A(x,t) = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -M^{-1}CK_I & -M^{-1}CK_P - K_I & -M^{-1}C - K_P \end{bmatrix} \qquad B(x,t) = \begin{bmatrix} 0 \\ 0 \\ M^{-1} \end{bmatrix}.$$

5. If any controller can stabilize the trajectory tracking system model (113), then it makes the original system (109) stable because the boundedness of a state vector $x$ implies those of $q$ and $\dot{q}$. However, the converse is not true.

$$|x| < \infty \quad \rightarrow \quad |e| < \infty \text{ and } |\dot{e}| < \infty \quad \rightarrow \quad |q| < \infty \text{ and } |\dot{q}| < \infty$$

6. For the set-point regulation control, the Lagrangian system model (109) can be rewritten by using the state vector $\dot{q}$ as follows:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = w_1(t, q) + \tau, \tag{114}$$

where $w_1(t, q) = -g(q) - d(t)$. On the other hand, for the trajectory tracking control, we obtained the system model (111) by using the composite state vector $s$. Here, we should notice that two system models Eqs. (111) and (114) show the same dynamic characteristics such as $-M(q)^{-1}C(q, \dot{q})$.

$$\ddot{q} = -M(q)^{-1}C(q, \dot{q})\dot{q} + M(q)^{-1}w_1(t, q) + M(q)^{-1}\tau \qquad \text{in terms of } \dot{q}$$
$$\dot{s} = -M(q)^{-1}C(q, \dot{q})s + M(q)^{-1}w(t, x) + M(q)^{-1}u \qquad \text{in terms of } s$$

## (PID) Class $\mathcal{K}$ function, class $\mathcal{K}_\infty$ function, and class $\mathcal{KL}$ function,

- class $\mathcal{K}$ function: $\gamma(|w|)$
- class $\mathcal{K}_\infty$ function: $\gamma(|x|)$
- class $\mathcal{KL}$ function: $\beta(|x|, t)$

# (PID) 5.2 $\mathcal{H}_\infty$ Optimality of PID Control

1. Among the stability theories, the notion of input-to-state stability (ISS) is more convenient to deal with the disturbance input than other theories.

2. (Basic Characteristics and Properties on the ISS)

   - the ISS notion is helpful to understand the effect of the set of inputs such as control, perturbation and disturbance on system states.
   - when the set of inputs go to zero, the behavior of system tends toward the equilibrium.
   - when there exist unknown bounded inputs such as perturbations and external disturbances acting on systems, the behavior of the system should remain bounded.

   $$|x(t)| \leq \beta(|x(0)|, t) + \gamma\left(\sup_{0\leq\tau\leq t} |w(\tau)|\right) \tag{115}$$

   where $\beta$ is class $\mathcal{KL}$ function and $\gamma$ is class $\mathcal{K}$ function.
   - as an alternative, the ISS of the control system can be proven by using Lyapunov function

   $$\dot{V} \leq -\gamma_1(|x|) + \gamma_2(|w|) \tag{116}$$

   where $\gamma_1$ and $\gamma_2$ are class $\mathcal{K}$ functions.
   - If $|x| \geq \rho(|w|)$ is satisfied, then we have

   $$|x| \geq \rho(|w|) \leftrightarrow \rho^{-1}(|x|) \geq |w| \quad \rightarrow \quad \dot{V} \leq -\gamma_1(|x|) + \gamma_2(\rho^{-1}(|x|)) \triangleq -\gamma_3(|x|) \tag{117}$$

   where $\rho$ and $\gamma_3$ are class $\mathcal{K}$ functions. In addition the global asymptotic stability is achieved only if $V(x) \rightarrow \infty$ as $x \rightarrow \infty$.

## (PID) 5.2.1 ISS-CLF for Lagrangian Systems

1. To show the ISS of the trajectory tracking system model, we should find both (1) the Lyapunov function and (2) the control law.

2. (Definition for ISS-CLF) A smooth positive definite radially unbounded function $V(x,t) : \Re^{3n} \times \Re_+ \to \Re_+$ is called an ISS-CLF (ISS Control Lyapunov Function) for the trajectory tracking system model, if $\exists$ a class $\mathcal{K}_\infty$ function $\rho$ such that the following implication holds for all $x \neq 0$ and all $w$:
$$|x| \geq \rho(|w|) \quad \Rightarrow \quad \inf_u \dot{V} < 0. \tag{118}$$

3. (Theorem 5.1) Let $s \triangleq \dot{e} + K_P e + K_I \int e dt \in \Re^n$. If the trajectory tracking system model of Eq. (113) is extended disturbance input-to-state stable (ISS), then the control law should have the following form with $\alpha \geq \frac{1}{2}$:
$$u = -\alpha K s - \rho^{-1}(|x|)\frac{s}{|s|}, \tag{119}$$
and $V(x,t) = \frac{1}{2}x^T P(x,t)x$ is an ISS-CLF with $\alpha = \frac{1}{2}$, where
$$P(x,t) = \begin{bmatrix} K_I M K_I + K_I K_P K & K_I M K_P + K_I K & K_I M \\ K_P M K_I + K_I K & K_P M K_P + K_P K & K_P M \\ M K_I & M K_P & M \end{bmatrix} \tag{120}$$
under the following two conditions for $P$:

   a) $K > 0, K_P > 0, K_I > 0$ constant diagonal matrices

   b) $K_P^2 > 2K_I$.

4. (Core of proof) From [HW # 8], we can get $\dot{V}$:

$$
\begin{aligned}
V_x B &= x^T P B = x^T [K_I, K_P, I]^T = s^T \\
V_t + V_x A x &= \frac{1}{2} x^T \left( \dot{P} + PA + A^T P \right) x = \frac{1}{2} x^T \begin{bmatrix} 0 & K_I K_P K & K_I K \\ K_I K_P K & 2K_I K & K_P K \\ K_I K & K_P K & 0 \end{bmatrix} x \\
&= \frac{1}{2} x^T \left( \begin{bmatrix} K_P^2 K & K_I K_P K & K_I K \\ K_I K_P K & K_P^2 K & K_P K \\ K_I K & K_P K & K \end{bmatrix} - \begin{bmatrix} K_I^2 K & 0 & 0 \\ 0 & (K_P^2 - 2K_I)K & 0 \\ 0 & 0 & K \end{bmatrix} \right) x \\
&= \frac{1}{2} x^T \begin{bmatrix} K_I \\ K_P \\ I \end{bmatrix} K \begin{bmatrix} K_I & K_P & I \end{bmatrix} x - \frac{1}{2} x^T \begin{bmatrix} K_I^2 K & 0 & 0 \\ 0 & (K_P^2 - 2K_I)K & 0 \\ 0 & 0 & K \end{bmatrix} x \\
&= \frac{1}{2} s^T K s - \frac{1}{2} x_1^T K_I^2 K x_1 - \frac{1}{2} x_2^2 (K_P^2 - 2K_I) K x_2 - \frac{1}{2} x_3^T K x_3 \\
\dot{V} &= V_t + V_x A x + V_x B w + V_x B u < 0 \\
&\Rightarrow \quad \frac{1}{2} s^T K s + s^T w + s^T u < \frac{1}{2} x_1^T K_I^2 K x_1 + \frac{1}{2} x_2^2 (K_P^2 - 2K_I) K x_2 + \frac{1}{2} x_3^T K x_3
\end{aligned}
$$

where the right-handed side of above inequality is positive definite under the given conditions $K > 0$, $K_P > 0$, $K_I > 0$, and $K_P^2 > 2K_I$.

If $|x| \geq \rho(|w|)$ and the controller (119) with $\alpha \geq \frac{1}{2}$ is applied, then the left-handed side of above inequality is negative definite

$$
\begin{aligned}
\frac{1}{2} s^T K s + s^T w + s^T u &\leq \frac{1}{2} s^T K s + |s||w| + s^T u && (121) \\
&\leq \frac{1}{2} s^T K s + |s| \rho^{-1}(|x|) + s^T u = -\left( \alpha - \frac{1}{2} \right) s^T K s < 0 && (122)
\end{aligned}
$$

Since $\inf_u \dot{V} < 0$ is achieved with $\alpha = \frac{1}{2}$, $V$ can be an ISS-CLF.

5. An important characteristics of controller (119) is that it has the PID control type as follows:

$$u = -\left(\alpha K + \frac{\rho^{-1}(|x|)}{|s|} I\right)\left(\dot{e} + K_P e + K_I \int e dt\right). \tag{123}$$

6. Another characteristics of above controller is that it can be rewritten as the optimal control type of

$$u = -R^{-1} B^T P x$$

by letting

$$R(x) \triangleq \left(\alpha K + \frac{\rho^{-1}(|x|)}{|s|} I\right)^{-1},$$

because $B^T P x = \dot{e} + K_P e + K_I \int e dt = s$.

## (PID) 5.2.2 $\mathcal{H}_\infty$ Optimality of PID Control Law

1. Now we are to show the $\mathcal{H}_\infty$ optimality of PID control type for the trajectory tracking systems by using the control law in Theorem 5.1.

2. Consider a general $\mathcal{H}_\infty$ performance index (PI) as following form:

$$J(t,x,u,w) = \lim_{t\to\infty}\left[2V(x(t),t) + \int_0^t \left(x^TQ(x)x + u^TR(x)u - \gamma^2 w^T w\right) d\sigma\right], \tag{124}$$

   where

   - $Q(x)$ is a state weighting matrix and $R(x)$ a control input weighting,
   - $\gamma$ means $\mathcal{L}_2$-gain.

3. (Refer to Eq. (104)) The HJI equation could be derived from the optimization for $\mathcal{H}_\infty$ performance index as following form:

$$HJI: \qquad \dot{P} + A^TP + PA - PBR^{-1}B^TP + \frac{1}{\gamma^2}PBB^TP + Q = 0. \tag{125}$$

4. (Theorem 5.2) For a given trajectory tracking system model (113), suppose that $\exists$ an ISS-CLF $V(t, x)$ in Theorem 5.1. If the PID control law (123) as following form:

$$u = -R^{-1}B^T P x \tag{126}$$

is utilized with conditions

a) $\alpha = 1$

b) $\rho^{-1}(|x|) \geq \frac{1}{\gamma^2}|s|$,

then the controller (126) is a solution of the minimization problem for $\mathcal{H}_\infty$ performance index (124) using

$$Q(x) = -\left(\dot{P} + A^T P + PA - PBKB^T P\right) \tag{127}$$

$$R(x) = \left(K + \frac{\rho^{-1}(|x|)}{|s|} I\right)^{-1}. \tag{128}$$

5. (Core of proof)

$$
\begin{aligned}
J &= \lim_{t\to\infty}\left[2V(x(t)) + \int_0^t (x^TQx + u^TRu - \gamma^2 w^Tw)\,d\tau\right] \\
&= \lim_{t\to\infty}\left[2V(x(t)) + \int_0^t x^T(-\dot{P} - A^TP - PA + PBKB^TP)x + u^TRu - \gamma^2 w^Tw\,d\tau\right] \\
&= \lim_{t\to\infty}\Bigg[2V(x(t)) - \int_0^t \left(x^T(\dot{P} + A^TP + PA)x + 2x^TPBu + 2x^TPBw\right)d\tau \\
&\quad + \int_0^t \left(x^TPBKB^TPx + 2x^TPBu + u^TRu\right)d\tau + \int_0^t \left(2x^TPBw - \gamma^2 w^Tw\right)d\tau\Bigg] \\
&= \lim_{t\to\infty}\Bigg[2V(x(t)) - 2\int_0^t \dot{V}d\tau + \int_0^t (u + R^{-1}B^TPx)^TR(u + R^{-1}B^TPx)d\tau \\
&\quad -\gamma^2\int_0^t (w - \frac{1}{\gamma^2}B^TPx)^T(w - \frac{1}{\gamma^2}B^TPx)d\tau - \int_0^t\left(\frac{\rho^{-1}(|x|)}{|s|} - \frac{1}{\gamma^2}\right)x^TPBB^TPxd\tau\Bigg] \\
&= 2V(x(0)) + \int_0^\infty (u + R^{-1}B^TPx)^TR(u + R^{-1}B^TPx)d\tau \\
&\quad -\gamma^2\int_0^\infty \left(w - \frac{1}{\gamma^2}s\right)^T\left(w - \frac{1}{\gamma^2}s\right)d\tau - \int_0^\infty\left(\frac{\rho^{-1}(|x|)}{|s|} - \frac{1}{\gamma^2}\right)|s|^2d\tau \\
&\le 2V(x(0))
\end{aligned}
$$

6. (Remark 5.1) The condition 2 in Theorem 5.2 is the design guideline of function $\rho^{-1}(|x|)$ to be $\mathcal{H}_\infty$ optimal controller. As a matter of fact, it implies that $\rho^{-1}(|x|)$ should not be smaller than at least the magnitude of worst case disturbance. Here, if we choose the magnitude of worst case disturbance for the function $\rho^{-1}(|x|)$, in other words, $\rho^{-1}(|x|) = |w^*| = \frac{1}{\gamma^2}|s|$, then the PID control law (126) recovers fortunately the static gain PID one because the matrix $R(x)$ of (128) becomes a constant matrix as follows:

$$R = \left(K + \frac{1}{\gamma^2} I\right)^{-1}. \tag{129}$$

7. In a viewpoint of an optimal control theory, the magnitude of a state weighting $Q$ has the relation with system errors

   - To reduce the error $|s|$, increase $K$. It will decrease $R$ and produce larger control input.
   - Notice that $K$ is common term in $Q$ and $R$.
   - $\mathcal{L}_2$-gain $\gamma$ has no effect on the state weighing $Q$, but it affects the control input weighting $R$. In other words, it does not affect the control performance by increasing the robustness to disturbances.

$$Q = \begin{bmatrix} K_I^2 K & 0 & 0 \\ 0 & (K_P^2 - 2K_I)K & 0 \\ 0 & 0 & K \end{bmatrix} \qquad R = \left(K + \frac{1}{\gamma^2} I\right)^{-1} \tag{130}$$

   - Approximately, we know that

$$K_I^2 K \propto \frac{1}{\left|\int e dt\right|^2} \qquad (K_P^2 - 2K_I)K \propto \frac{1}{|e|^2} \qquad K \propto \frac{1}{|\dot{e}|^2} \qquad R \propto \frac{1}{|u|^2}$$

# (PID) 5.3 Inverse Optimal PID Control

1. In the previous section, the $\mathcal{H}_\infty$ optimality of PID controller for the performance index was shown through Theorems 5.1, 5.2 and Remark 5.1. Here, we define the inverse optimal PID controller using the static gain one in Remark 5.1 and summarize its design conditions in following Theorem.

2. (Theorem 5.3) If the inverse optimal PID controller:

$$\tau = \left(K + \frac{1}{\gamma^2} I\right)\left(\dot{e} + K_P e + K_I \int e dt\right) \tag{131}$$

   satisfying next conditions:

   a) $K > 0, K_P > 0, K_I > 0$, constant diagonal matrices
   b) $K_P^2 > 2K_I$,
   c) $\gamma > 0$

   is applied to the trajectory tracking system model (113), then the closed-loop control system is extended disturbance input-to-state stable(ISS).

3. (Core of proof)

$$\begin{aligned}\dot{V} &= V_t + V_x A x + V_x B u + V_x B w \\ &= \frac{1}{2} x^T (\dot{P} + A^T P + PA) x - x^T PBR^{-1} BPx + x^T PBw\end{aligned}$$

If Young's inequality $x^T PBw \leq \frac{1}{\gamma^2} x^T PBB^T P + \gamma^2 |w|^2$ is applied, then we have

$$\begin{aligned}\dot{V} &\leq \frac{1}{2} x^T (\dot{P} + A^T P + PA) x - x^T PBR^{-1} BPx + \frac{1}{\gamma^2} x^T PBB^T P + \gamma^2 |w|^2 \\ &= -\frac{1}{2} x^T (Q - PBKBP) x - x^T PBR^{-1} BPx + \frac{1}{\gamma^2} x^T PBB^T P + \gamma^2 |w|^2 \\ &= -\frac{1}{2} x^T Q x + \frac{1}{2} x^T PBKBPx - x^T PBKBPx + \gamma^2 |w|^2 \\ &= -\frac{1}{2} x^T (Q + PBKB^T P) x + \gamma^2 |w|^2 \end{aligned} \tag{132}$$

since

$$K = R^{-1} - \frac{1}{\gamma^2} I$$

4. (Corollary 5.1) The inverse optimal PID controller of (131) exists if and only if the trajectory tracking system (113) is extended disturbance input-to-state stable (ISS).

5. (Core of proof)

$$
\begin{array}{ccc}
\text{ISS} & \xrightarrow{\text{Theorem 5.1}} & \text{Control Law (119)} \\
\text{Theorem 5.3} \Big\uparrow & & \Big\downarrow \text{Theorem 5.2} \\
\text{Inverse Optimal PID (131)} & \xleftarrow[\text{Remark 5.1}]{} & \mathcal{H}_\infty \text{ Optimality}
\end{array}
$$

# (PID) 5.4 Selection Guidelines for Gains

1. Let us reconsider the extended disturbance of (110) as follows:

$$
\begin{aligned}
|w|^2 &= \left| M\left(\ddot{q}_d + K_P \dot{e} + K_I e\right) + C\left(\dot{q}_d + K_P e + K_I \int e dt\right) + g + d \right|^2 \\
&= \left| M K_P \dot{e} + M K_I e + C s - C \dot{e} + h \right|^2 \quad \text{(by Schwarz inequality)} \\
&\leq 5\left| M K_P \dot{e} \right|^2 + 5\left| C \dot{e} \right|^2 + 5\left| M K_I e \right|^2 + 5\left| C s \right|^2 + 5\left| h \right|^2, \qquad (133)
\end{aligned}
$$

where $h = M(q)\ddot{q}_d + C(q, \dot{q})\dot{q}_d + g(q) + d(t)$.

2. (Theorem 5.4) Let $|M(q)| \leq m$, $|C(q, \dot{q})| \leq c_0|\dot{q}|$, $K = kI, K_P = k_P I$ and $K_I = k_I I \in \Re^{n\times n}$. Suppose that the tuning variables $(\gamma, k)$ satisfy following condition

$$
\frac{\sqrt{k}}{\gamma} > \sqrt{10} c_0 |\dot{q}|, \qquad \Rightarrow \qquad \frac{\sqrt{k}}{\gamma} \propto \max(\dot{q}) \qquad (134)
$$

then the gain $k_P$ should be confined to the following constraint:

$$
k_P < \frac{\sqrt{(k/\gamma^2) - 10 c_0^2 |\dot{q}|^2}}{\sqrt{10} m}, \qquad \Rightarrow \qquad k_P \propto \frac{1}{m} \frac{\sqrt{k}}{\gamma} \qquad (135)
$$

and the gain $k_I$ should be confined to the following constraint:

$$
k_I < \frac{\sqrt{(k/\gamma^2)^2 + 10 m^2 k_P^2 (k/\gamma^2)} - k/\gamma^2}{10 m^2}. \qquad \Rightarrow \qquad k_I \propto \frac{k_P}{m} \frac{\sqrt{k}}{\gamma} \qquad (136)
$$

3. (Core of proof)

$$
\begin{aligned}
\dot{V} &\leq -\frac{1}{2}x^T(Q + PBKB^TP)x + \gamma^2|w|^2 \\
&= -\frac{1}{2}\left(x_1^T K_I^2 K x_1 + x_2^2(K_P^2 - 2K_I)Kx_2 + x_3^T K x_3\right) - \frac{1}{2}s^T K s \\
&\quad + \gamma^2\left(5\left|MK_P\dot{e}\right|^2 + 5\left|C\dot{e}\right|^2 + 5\left|MK_I e\right|^2 + 5\left|Cs\right|^2 + 5\left|h\right|^2\right) \\
&\leq -\frac{1}{2}\left(kk_I^2|\int edt|^2 + k(k_P^2 - 2k_I)|e|^2 + k|\dot{e}|^2\right) - \frac{1}{2}k|s|^2 \\
&\quad + \gamma^2\left(5m^2k_P^2\left|\dot{e}\right|^2 + 5c_0^2|\dot{q}|^2\left|\dot{e}\right|^2 + 5m^2k_I^2\left|e\right|^2 + 5c_0^2|\dot{q}|^2|s|^2 + 5\left|h\right|^2\right) \\
&= -\frac{1}{2}(k - 10\gamma^2 m^2 k_P^2 - 10\gamma^2 c_0^2|\dot{q}|^2)|\dot{e}|^2 \qquad \Rightarrow \qquad k_P < \frac{\sqrt{(k/\gamma^2) - 10c_0^2|\dot{q}|^2}}{\sqrt{10}m} \\
&\quad - \frac{1}{2}(k(k_P^2 - 2k_I) - 10\gamma^2 m^2 k_I^2)|e|^2 \qquad \Rightarrow \qquad k_I < \frac{\sqrt{(k/\gamma^2)^2 + 10m^2k_P^2(k/\gamma^2)} - k/\gamma^2}{10m^2} \\
&\quad - \frac{1}{2}kk_I^2|\int edt|^2 \\
&\quad - \frac{1}{2}(k - 10\gamma^2 c_0^2|\dot{q}|^2)|s|^2 \qquad \Rightarrow \qquad \frac{\sqrt{k}}{\gamma} > \sqrt{10}c_0|\dot{q}| \\
&\quad + 5\gamma^2|h|^2
\end{aligned}
$$

- (HW # 9) solve problems 5.1 and 5.2