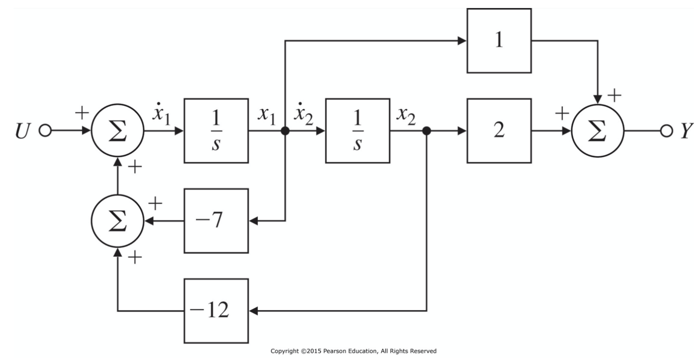
4 Analysis of the State Equations

- (7.4.1) Block Diagrams and Canonical Forms

   1. Control canonical form shown in Fig. 7.7 has a feature that each state-variable feeds back to the control input u, through the coefficients of the system matrix A_c. Consider the following



TF:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12} \quad \rightarrow \quad G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12}\frac{X(s)}{X(s)}$$

From above, we know that

$$Y(s) = (s+2)X(s) \qquad y = \dot{x} + 2x$$

$$U(s) = (s^2+7s+12)X(s) \qquad u = \ddot{x} + 7\dot{x} + 12x$$

Let us assign the states $x_1 = \dot{x}$ and $x_2 = x$ as follows:

$$
\begin{aligned}
y &= x_1 + 2x_2 \\
\dot{x}_2 &= \dot{x} = x_1 \\
\dot{x}_1 &= -7\dot{x} - 12x + u = -7x_1 - 12x_2 + u
\end{aligned}
$$

These three equations can then be rewritten in the matrix form:

$$
\begin{aligned}
\dot{x} &= A_c x + B_c u \\
y &= C_c x + D_c x
\end{aligned}
$$

where

$$
A_c = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C_c = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad D_c = 0
$$

where the subscript c refers to control canonical form.

General control canonical form: consider the following TF

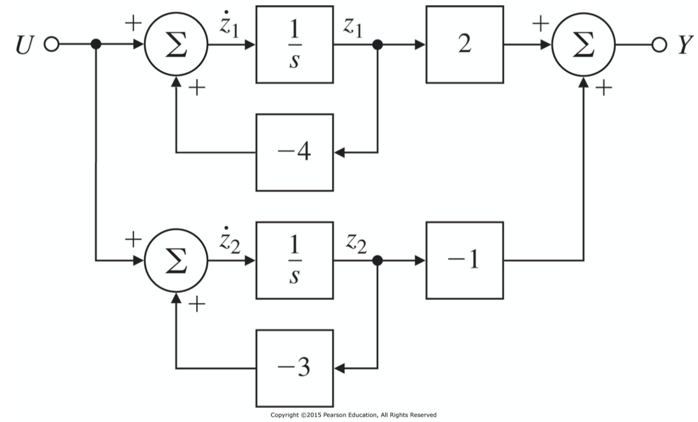
$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 + s^{n-2} + \cdots + a_n}$$

Then we have

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} b_1 & b_2 & \cdots & \cdots & b_n \end{bmatrix} \qquad D_c = 0$$

2. Modal canonical form shown in Fig 7.8 has a feature that the system poles appear in the diagonal of the system matrix A_m. Consider the TF



$$G(s) = \frac{s+2}{s^2+7s+12} = \frac{2}{s+4} + \frac{-1}{s+3} \quad \rightarrow \quad G = 2G_1 + (-1)G_2$$

$$G_1 = \frac{Z_1(s)}{U(s)} = \frac{1}{s+4} \quad \text{and} \quad G_2 = \frac{Z_2(s)}{U(s)} = \frac{1}{s+3}$$

let us obtain the state-space equations:

$$y = 2z_1 - z_2$$

$$\dot{z}_1 = -4z_1 + u$$

$$\dot{z}_2 = -3z_2 + u$$

then the modal canonical form is easily obtained as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \qquad \dot{z} = A_m z + B_m u$$

$$y = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + 0 \cdot u \qquad y = C_m z + D_m u$$

where the subscript m refers to modal canonical form. Modes represent the poles (or eigenvalues) of the given system. Note that the complex poles appear along the diagonal with off-diagonal terms indicating the coupling.

3. (Example) Find the modal form of the TF

$$G(s) = \frac{2s+4}{s^2(s^2+2s+4)}$$

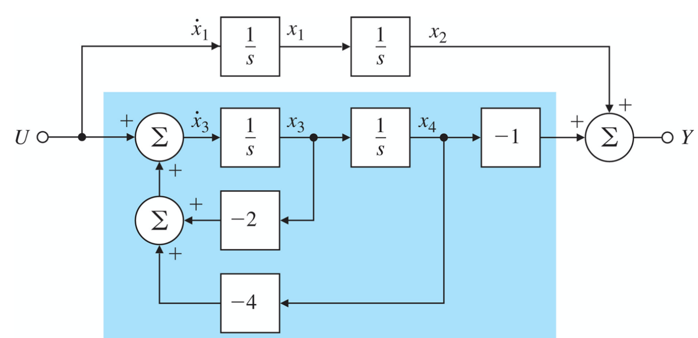
By taking the partial fraction method, we have

$$G(s) = \frac{1}{s^2} + \frac{-1}{s^2+2s+4} \quad \rightarrow \quad G = G_1 + (-1)G_2 \quad \rightarrow \quad G_1 = \frac{Z_2(s)}{U(s)} = \frac{1}{s^2} \text{ and } G_2 = \frac{Z_4(s)}{U(s)} = \frac{1}{s^2+2s+4}$$

let us obtain the state-space equations:

$$\begin{aligned} y &= z_2 - z_4 \\ \dot{z}_1 &= u \\ \dot{z}_2 &= z_1 \\ \dot{z}_3 &= -2z_3 - 4z_4 + u \\ \dot{z}_4 &= z_3 \end{aligned}$$

then the modal form is obtained as shown in the following figure



$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u \qquad \dot{z} = A_m z + B_m u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + 0 \cdot u \qquad y = C_m z + D_m u$$

4. (Example 7.8)

5. Consider a system described by the state equation:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where it is noted that the state space description is not unique. For a nonsingular transformation matrix T, if we let

$$x = Tz \quad \rightarrow \quad \dot{x} = T\dot{z} \quad \rightarrow \quad z = T^{-1}x$$

then we have

$$T\dot{z} = ATz + Bu \qquad \dot{z} = \overline{A}z + \overline{B}u$$
$$y = CTz + Du \qquad y = \overline{C}z + \overline{D}u$$

where

$$\overline{A} = T^{-1}AT \qquad \overline{B} = T^{-1}B \qquad \overline{C} = CT \qquad \overline{D} = D$$

In order to find the control canonical form, first, let us consider the following equation:

$$\overline{A}T^{-1} = T^{-1}A$$

If $\overline{A}$ is in control canonical form, and we describe T^{-1} as a matrix with row vectors t_1, t_2, t_3, then

$$\begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} A = \begin{bmatrix} t_1 A \\ t_2 A \\ t_3 A \end{bmatrix} \quad \rightarrow \quad \therefore \quad t_2 = t_3 A \quad \text{and} \quad t_1 = t_2 A = t_3 A^2$$

second,

$$T^{-1} B = \overline{B} \quad \rightarrow \quad \begin{bmatrix} t_1 B \\ t_2 B \\ t_3 B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \therefore \quad t_3 B = 0 \quad t_2 B = t_3 A B = 0 \quad t_1 B = t_3 A^2 B = 1$$

These equations can, in turn, be written in matrix form as

$$t_3 \begin{bmatrix} B & AB & A^2 B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad t_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}$$

where the controllability matrix $\mathcal{C} = \begin{bmatrix} B & AB & A^2 B \end{bmatrix}$. And furthermore, since $t_2 = t_3 A$ and $t_1 = t_3 A^2$, we can construct all the rows of T^{-1}

- When the controllability matrix $\mathcal{C}$ is nonsingular, the corresponding A and B matrices are said to be controllable.
- One can always transform a given state description to control canonical form if and only if the controllability matrix $\mathcal{C}$ is nonsingular.
- Pole-zero cancellation in the TF brings that the controllability loses its rank.
- Controllability is a function of the state of the system and cannot be decided from a TF.

6. Observer canonical form is obtained by transposing the system matrix

$$A_o = A_c^T \qquad B_o = C_c^T \qquad C_o = B_c^T \qquad D_o = D_c$$

for example, from the following control canonical form,

$$\dot{x} = A_c x + B_c u$$
$$y = C_c x + D_c u$$

where

$$A_c = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C_c = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad D_c = 0$$

we can get the observer canonical from as follows:

$$\dot{x} = A_o x + B_o u$$
$$y = C_o x + D_o u$$

where

$$A_o = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \qquad B_o = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad C_o = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D_o = 0$$

7. In order to find the modal form, we assume that the system has only distinct poles. Consider

$$T\overline{A} = AT$$

$$\begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} = A \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}$$

where t_1, t_2, t_3 are column vectors of the transformation matrix T. Above equation is equivalent to the three vector-matrix equations:

$$p_i t_i = A t_i \qquad i = 1, 2, 3 \qquad \rightarrow \qquad (A - p_i I) t_i = 0$$

where it is equal to the eigenvalue and eigenvector problems. In other words, p_i is given eigenvalues and t_i is a corresponding eigenvector.

8. (Example 7.9) Find the matrix to transform the control canonical form of Eq. 7.12 into the modal form of Eq. 7.14 in the textbook?

$$A_c = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C_c = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad D_c = 0$$

$$A_m = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} \qquad B_m = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad C_m = \begin{bmatrix} 2 & 1 \end{bmatrix} \qquad D_m = 0$$

For $p_1 = -4$,

$$-4 \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix} \qquad \rightarrow \qquad t_{11} = -4t_{21}$$

For $p_1 = -3$,

$$-3 \begin{bmatrix} t_{12} \\ t_{22} \end{bmatrix} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} t_{12} \\ t_{22} \end{bmatrix} \qquad \rightarrow \qquad t_{12} = -3t_{22}$$

If we choose $t_{21} = -1$ and $t_{22} = 1$, then we have

$$T = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

Thus we can know that

$$A_m = T^{-1} A_c T \qquad B_m = T^{-1} B_c$$

$$C_m = C_c T \qquad D_m = D_c$$

9. (Example 7.10)