9 Introduction of the Reference Input with the Estimator

- Good *disturbance rejection* and good *command following* need to be taken into account in designing a control system.

- Let us consider the plant and controller equations for the full-order estimator:

$$\begin{array}{lll} \text{Plant} & \dot{x} = Ax + Bu & y = Cx \\ \text{Controller} & \dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly & u = -K\hat{x} \end{array}$$

- (7.9.1) General Structure for the Reference Input

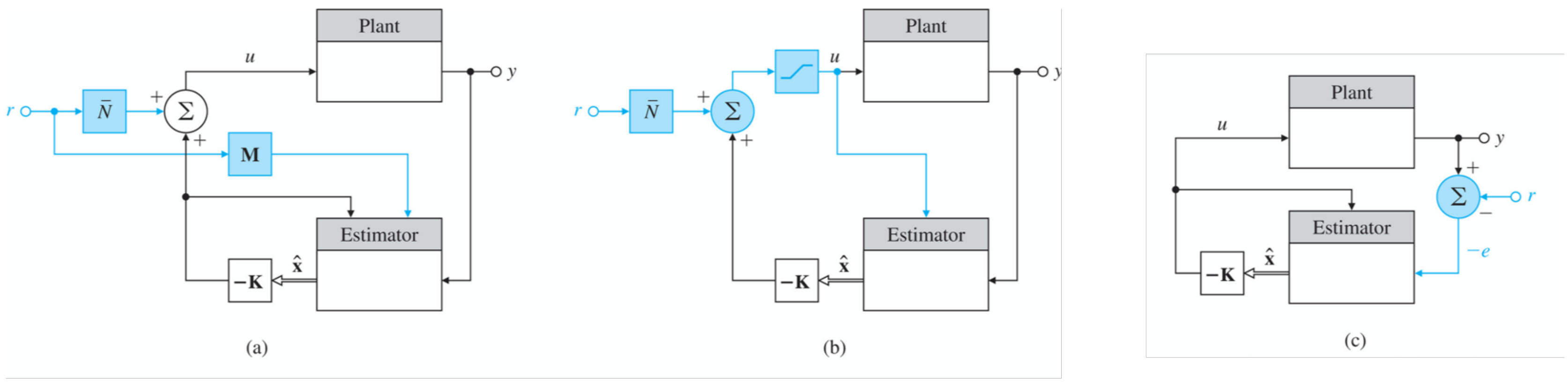
1. Given a reference input $r(t)$, the most general linear way to introduce r into the system equations is to add terms proportional to it in the controller equations. Let us add $\bar{N}r$ and Mr as follows:

$$\text{Controller} \quad \dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly + Mr \qquad u = -K\hat{x} + \bar{N}r$$

   where $\bar{N}$ is a scalar and M is an $n \times 1$ vector

2. It is clear that neither M nor $\bar{N}$ affects the characteristic equation of the combined controller-estimator system. In the TF from r to y, the selection of M and $\bar{N}$ will affect only the *zeros of transmission from r to y*. As a consequence, it can affect the transient response but not the stability.

3. There are three strategies for choosing M and $\bar{N}$:



a) (Autonomous estimator) Select M and $\bar{N}$ so that the state estimator error is independent of r.

– Estimation error equation becomes

$$\dot{x} - \dot{\hat{x}} = Ax + B[-K\hat{x} + \bar{N}r] - [(A - BK - LC)\hat{x} + Ly + Mr]$$
$$\dot{\tilde{x}} = (A - LC)\tilde{x} + B\bar{N}r - Mr$$

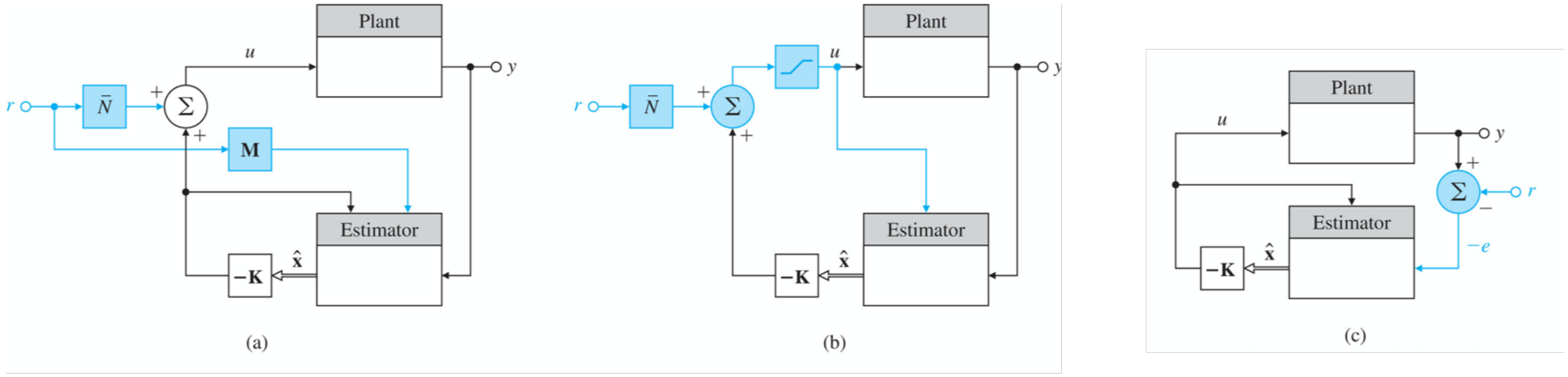
– If r is not to appear in the above, then we should choose

$$\therefore \qquad M = B\bar{N}$$

– A a result,

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly \qquad\qquad u = -K\hat{x} + \bar{N}r$$

for practical use, the actuator saturation should be considered as shown in the figure 7.48(b).



b) (Tracking error estimator) Select M and $\bar{N}$ so that only the tracking error $e = r - y$ is used in the control as shown in figure 7.48(c).

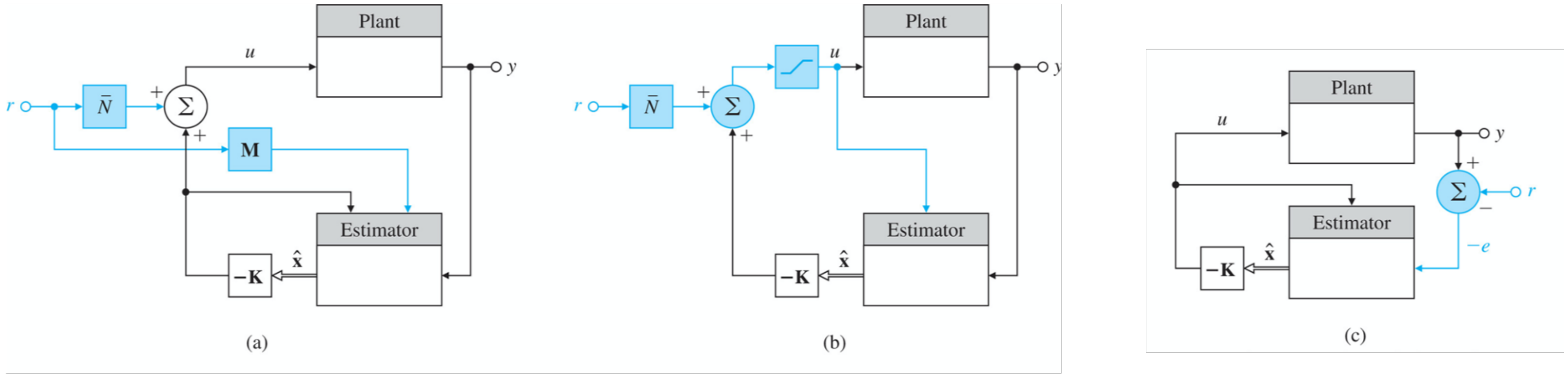
- The requirement is satisfied if we select

$$\bar{N} = 0 \quad \text{and} \quad M = -L$$

- Then the estimator equation is

$$\dot{\hat{x}} = (A - BK - LC)\hat{x} + L(y - r) \qquad\qquad u = -K\hat{x}$$

- The compensator in this case is a standard lead compensator in the forward path.



c) (Zero assignment estimator) Select M and $\bar{N}$ so that n of the zeros of the overall TF are assigned at places of the designer's choice.

- Reconsider the general form shown in figure 7.48(a)
- From the general form of controller, the equation for a zero from r to u with $y = 0$ is given by

$$\det \begin{bmatrix} sI - A + BK + LC & -M \\ -K & \bar{N} \end{bmatrix} = \det \begin{bmatrix} sI - A + BK + LC & -\frac{1}{\bar{N}}M \\ -K & 1 \end{bmatrix}$$

$$= \det \begin{bmatrix} sI - A + BK + LC - \frac{1}{\bar{N}}MK & -\frac{1}{\bar{N}}M \\ 0 & 1 \end{bmatrix}$$

$$\gamma(s) = \det \left[sI - A + BK + LC - \frac{1}{\bar{N}}MK \right] = 0$$

- Here we have to select $\frac{1}{\bar{N}}M$ for a desired zero polynomial $\gamma(s)$ in the TF from r to u
- The zeros influence the transient response significantly

d) (Truxal's formula) If the system is Type 1, then the steady-state error to a step input will be zero and to a unit-ramp input will be

$$e_{ss} = \frac{1}{K_v} = \sum \frac{1}{z_i} - \sum \frac{1}{p_i}$$

where K_v is the velocity constant, p_i denotes the closed-loop poles and z_i the closed-loop zeros.

- If $\|z_i - p_i\| \ll 1$, then the effect of this pole-zero pair on the dynamic response will be small
- Even though $z_i - p_i$ is small, it is possible for $\frac{1}{z_i} - \frac{1}{p_i}$ to be substantial and thus to have a significant influence on K_v.
- Application of above two guidelines to the selection of $\gamma(s)$, and hence of M and $\bar{N}$, results in a lag-network design.

4. (Example 7.33) Design a controller using pole placement so that both poles are at $s = -2 \pm j2$ and the system has a velocity constant $K_v = 10$.

$$G(s) = \frac{1}{s(s+1)} \qquad \Rightarrow \qquad \begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

The feedback gain $K = [k_1 \ k_2]$ of $u = -Kx$ is

$$\det[sI - A + BK] = \det \begin{bmatrix} s & -1 \\ k_1 & s+1+k_2 \end{bmatrix} = s^2 + (1+k_2)s + k_1 = \alpha_c(s) = s^2 + 4s + 8 \quad \rightarrow \quad K = [8 \ 3]$$

Let us apply the zero placement along with Truxal's formula.

a) First we must select the estimator pole p_3 and the zero z_3 for $K_v = 10$. Also, we want to keep $z_3 - p_3$ small so that there is little effect on the dynamic response, and yet $\frac{1}{z_3} - \frac{1}{p_3}$ be large enough to increase the value of K_v.

b) For example, $p_3 = -0.1$, let us solve the following:

$$\begin{aligned} \frac{1}{K_v} &= \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} \\ &= \frac{1}{\infty} + \frac{1}{\infty} + \frac{1}{z_3} - \frac{1}{-2+2j} - \frac{1}{-2-2j} - \frac{1}{-0.1} = \frac{1}{10} \\ &= 0 + 0 + \frac{1}{z_3} - \frac{-2-2j}{8} - \frac{-2+2j}{8} + 10 = 0.1 \end{aligned}$$

Thus we have

$$\frac{1}{z_3} = -9.9 - \frac{1}{2} = -10.4 \qquad \rightarrow \qquad z_3 = -0.096$$

c) Finally, we can design the overall TF as follows:

$$\frac{Y(s)}{R(s)} = \frac{8.32(s + 0.096)}{(s^2 + 4s + 8)(s + 0.1)}$$

5. Let us summarize our findings on the effect of introducing the reference input. When the reference input signal is included in the controller, the overall TF of the closed-loop system is

$$T(s) = \frac{Y(s)}{R(s)} = K_s \frac{\gamma(s)b(s)}{\alpha_c(s)\alpha_e(s)}$$

where

- K_s is the total system gain
- $\alpha_c(s)$ results in a control gain K such that $\det[sI - A + BK] = \alpha_c(s)$
- $\alpha_e(s)$ results in an estimator gain L such that $\det[sI - A + LC] = \alpha_e(s)$
- $\gamma(s)$ is a zero polynomial to be designed by M and $\bar{N}$
- $b(s)$ is the plant zeros ($G(s) = \frac{b(s)}{a(s)}$) which are not moved by this technique and remain as part of the closed-loop TF.

- (7.9.2) Selecting the Gain

$$\text{Plant} \quad \dot{x} = Ax + Bu \qquad\qquad y = Cx$$

$$\text{Controller} \quad \dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly + Mr \qquad\qquad u = -K\hat{x} + \bar{N}r$$

1. If we choose a first method (autonomous estimator), with $\bar{N} = N_u + KN_x$ and $M = B\bar{N}$, the control is given by

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly \qquad\qquad u = -K\hat{x} + \bar{N}r = N_u r - K(\hat{x} - N_x r)$$

   This is the most common choice.

2. If we use the second method (tracking error estimator), the result is trivial; recall that $\bar{N} = 0$ and $M = -L$ for the error control.

$$\dot{\hat{x}} = (A - BK - LC)\hat{x} + L(y - r) \qquad\qquad u = -K\hat{x}$$

3. If we use the third method (zero assignment estimator), we pick $\bar{N}$ such that the overall closed-loop DC gain is unity. The closed-loop system becomes

$$\dot{x} = Ax - BK\hat{x} + B\bar{N}r = (A - BK)x + BK\tilde{x} + B\bar{N}r$$
$$\dot{\tilde{x}} = Ax - BK\hat{x} + B\bar{N}r - (A - BK - LC)\hat{x} - LCx - Mr = (A - LC)\tilde{x} + (B\bar{N} - M)r$$
$$y = Cx$$

In other expression, we have

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ B - \frac{M}{\bar{N}} \end{bmatrix} \bar{N} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

where $\frac{M}{\bar{N}}$ is the outcome of selecting zero locations from $\gamma(s)$. The closed-loop system has unity DC gain if

$$\lim_{s \to 0} \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} sI - A + BK & -BK \\ 0 & sI - A + LC \end{bmatrix}^{-1} \begin{bmatrix} B \\ B - \frac{M}{\bar{N}} \end{bmatrix} \bar{N} = 1$$

If we solve the above, we have

$$\bar{N} = -\frac{1}{C(A - BK)^{-1} B[1 - K(A - LC)^{-1}(B - \frac{M}{\bar{N}})]}$$

where $\frac{1}{\bar{N}} M$ is chosen from the desired zero polynomial

$$\gamma(s) = \det\left(sI - A + BK + LC - \frac{M}{\bar{N}} K\right)$$