제 2 장

Dynamic Models

1 Dynamics of Mechanical Systems

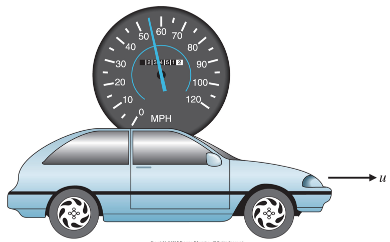
1. Translational Motion

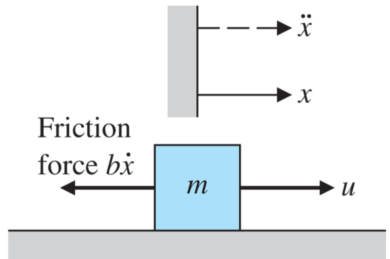
   - Newton's 2nd law

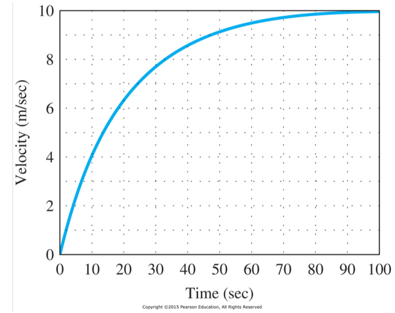
     $$\sum F = m \cdot a$$

     vector sum of all forces applied to each body $=$ mass of the body $\cdot$ vector acceleration of each body

   - A force of $1N$ will impart an acceleration of $1m/s^2$ to a mass of $1kg$.
   - The weight of an object is mg, where g is the acceleration of gravity ($= 9.81m/s^2$), which is the quantity measured on scales.







- (Example 2.1, Cruise Control Model) Input jumps from being $u = 0$ at time $t = 0$ to a constant $u = 500N$, hereafter, assume that $m = 1000kg$, viscous drag coefficient $b = 50Ns/m$, and initial speed $v(0) = 0$. From the free-body diagram,

$$u - bv = m\dot{v}$$

In order to solve above equation, let us take Laplace transform (LT) explained in the next chapter like these: $\mathcal{L}[u(t)] = U(s)$, $\mathcal{L}[v(t)] = V(s)$ and $\mathcal{L}[\dot{v}(t)] = sV(s) - v(0)$

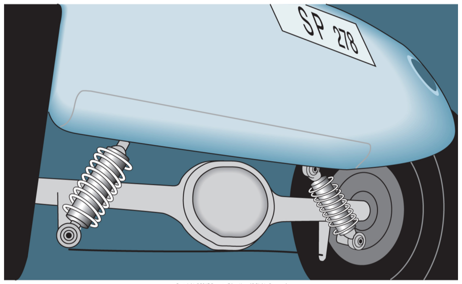
$$U(s) - bV(s) = m[sV(s) - v(0)] \qquad \rightarrow \qquad \frac{V(s)}{U(s)} = \frac{1}{ms+b} = \frac{1}{1000s+50} = \frac{0.001}{s+0.05}$$

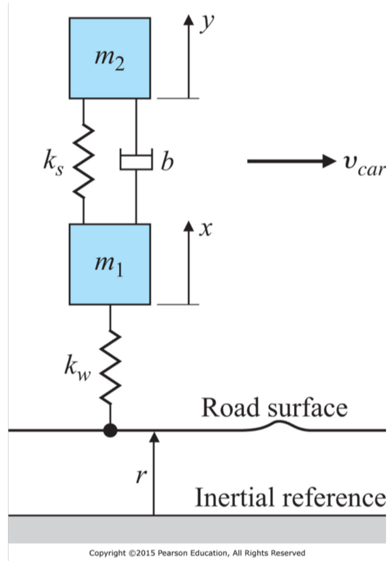
Also, since the LT of unit-step function is $\frac{1}{s}$, we can get

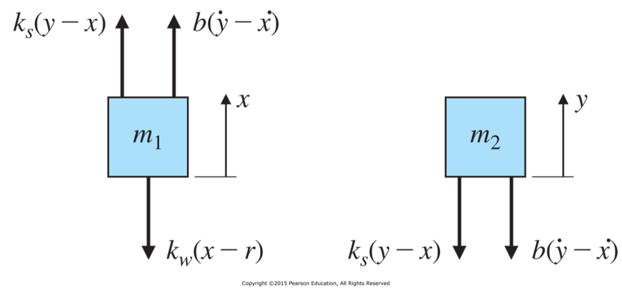
$$V(s) = \frac{0.001}{s+0.05} \cdot \frac{500}{s} = \frac{10}{s} - \frac{10}{s+0.05}$$

Now let us take inverse Laplace transform (iLT) to get the time-domain profile of speed by using $\mathcal{L}^{-1}[\frac{1}{s}] = 1$ and $\mathcal{L}^{-1}[\frac{1}{s+a}] = e^{-at}$ for $t \geq 0$ as follow:

$$\therefore \qquad v(t) = 10 - 10e^{-0.05t} \quad \text{for} \quad t \geq 0$$







- (Example 2.2, Two-Mass System, Suspension Model on Bumpy Road) Input is the bumpy road $r(t)$ and output is the position of upper mass (seat) y.

$$m_1\ddot{x} = -k_s(x-y) - b(\dot{x}-\dot{y}) - k_w(x-r) \qquad m_2\ddot{y} = -k_s(y-x) - b(\dot{y}-\dot{x})$$

Let us take LT with zero initial conditions, then we have

$$(m_1s^2 + bs + (k_s+k_w))X(s) = (bs+k_s)Y(s) + k_wR(s) \qquad (m_2s^2 + bs + k_s)Y(s) = (bs+k_s)X(s)$$

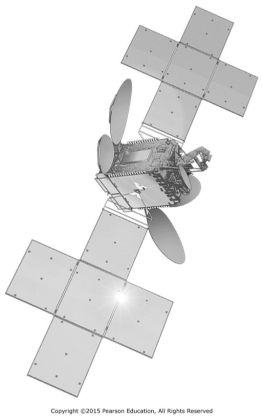
After tedious manipulation, we can get the relation of $\frac{Y(s)}{R(s)}$

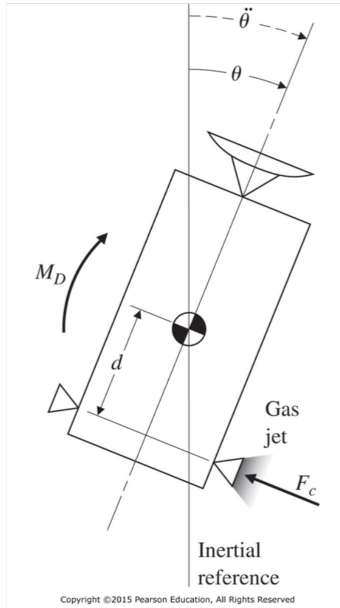
2. Rotational Motion

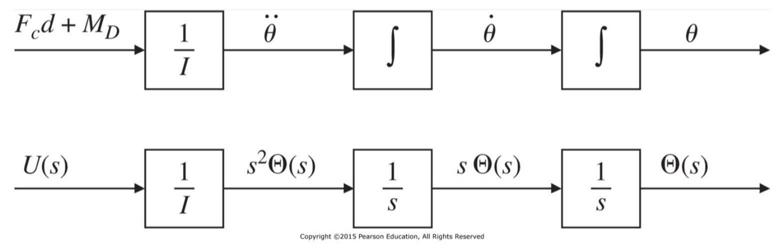
- Euler's law

$$\sum M = I \cdot \alpha$$

$$\text{sum of all moments} = \text{body's mass moment of inertia} \cdot \text{angular acceleration}$$







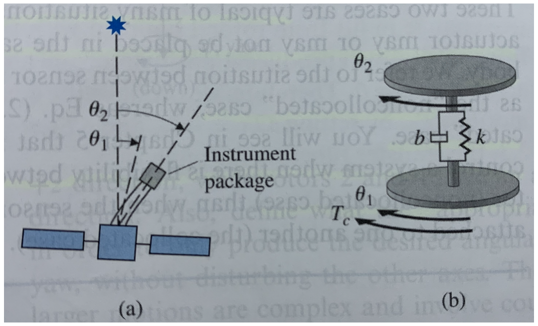
- (Example 2.3, Satellite Attitude Control Model) Reaction jets produce a moment of $F_c d$ about the mass center and there is small disturbance moment M_D on the satellite caused from the solar pressure acting on any asymmetric solar panels. From the free-body diagram,

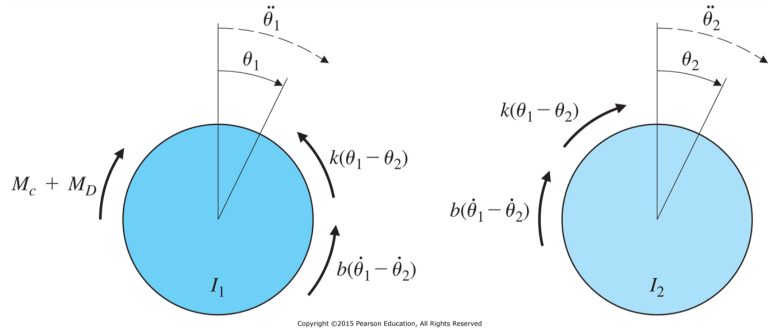
$$F_c d + M_D = I\ddot{\theta}$$

with zero initial conditions of $\theta(0) = 0$ and $\dot{\theta}(0) = 0$, then we have a Transfer Function (TF) as follow:

$$\frac{\Theta(s)}{U(s)} = \frac{1}{Is^2}$$

where $U(s) = \mathcal{L}(F_c d + M_D)$ and above system is referred to as "double integrator plant".





- (Example 2.4, Flexible Satellite Attitude Control) Particular difficulty arises when there is flexibility b/w the sensor and actuator locations. Let us denote the applied input torque as $T_c = M_c$ and the disturbance torque as $M_D = 0$. From the free-body diagram,

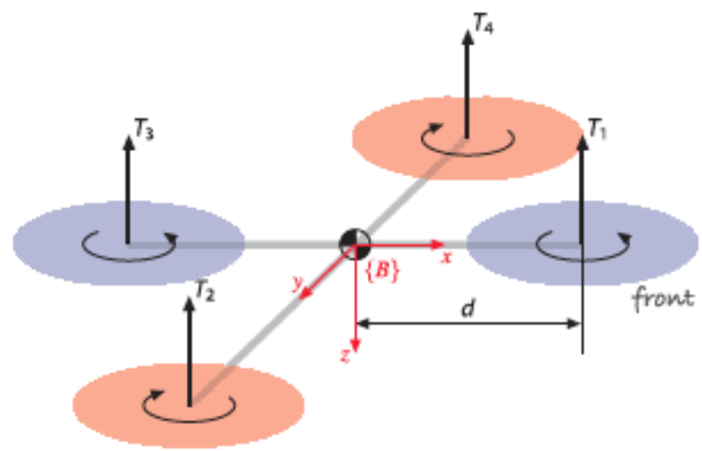
$$I_1\dot{\theta}_1 = T_c - b(\dot{\theta}_1 - \dot{\theta}_2) - k(\theta_1 - \theta_2)$$
$$I_2\dot{\theta}_2 = -b(\dot{\theta}_2 - \dot{\theta}_1) - k(\theta_2 - \theta_1)$$

For simplicity, ignoring the damping b,

$$(I_1 s^2 + k)\Theta_1(s) = T_c(s) + k\Theta_2(s)$$
$$(I_2 s^2 + k)\Theta_2(s) = k\Theta_1(s)$$

we have LT under zero initial conditions as follows:

$$\frac{\Theta_1(s)}{T_c(s)} = \frac{I_2 s^2 + k}{(I_1 s^2 + k)(I_2 s^2 + k) - k^2} \qquad \text{collocated case}$$
$$\frac{\Theta_2(s)}{T_c(s)} = \frac{k}{(I_1 s^2 + k)(I_2 s^2 + k) - k^2} \qquad \text{non-collocated case}$$



- (Example 2.5, Quadrotor Drone) The body-fixed coordinate frame B has its z-axis downward following the aerospace convention. The quadrotor has four rotors, labeled 1 to 4, mounted at the end of each cross arm.
  - The rotor speed is ω_i and the thrust is an upward vector

$$T_i = b\omega_i^2 \qquad \text{for} \quad i = 1, 2, 3, 4$$

    in the drone's negative z-direction, where b is a lift constant. $T = \sum T_i$ is the total upward thrust.
  - Pairwise differences in rotor thrusts cause the drone to rotate. Rolling torque about x-axis and pitching torque about y-axis are

$$\tau_x = d(T_4 - T_2) = db(\omega_4^2 - \omega_2^2) \qquad \text{and} \qquad \tau_y = d(T_1 - T_3) = db(\omega_1^2 - \omega_3^2))$$

    where d is the distance from the motor to the mass center.
  - The torque applied to each propeller by the motor is a reaction torque on the airframe which acts to rotate the airframe about the propeller shaft in the opposite direction to

its rotation.

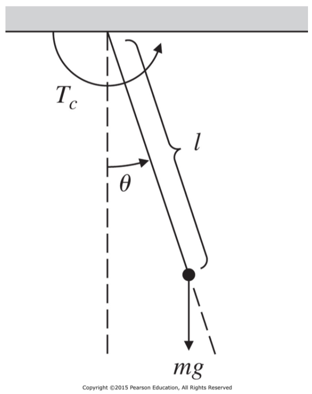
$$Q_i = k\omega_i^2$$

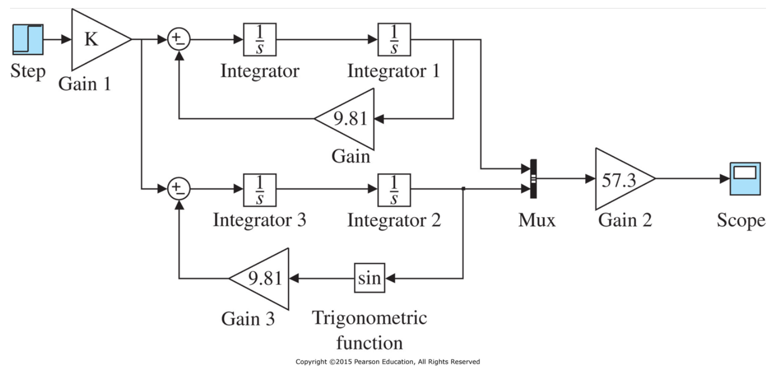
where k depends on the same factors as b.

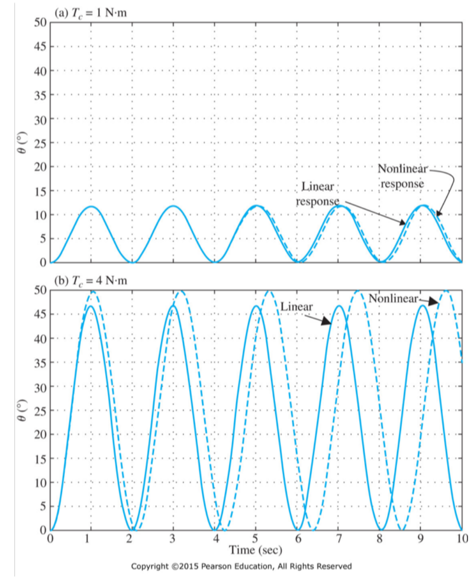
- The total reaction torque about the z-axis is

$$\tau_z = Q_1 - Q_2 + Q_3 - Q_4 = k(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2)$$

where the different signs are due to the different rotation directions of the rotors.







- (Example 2.6, 2.7, Pendulum) Assume that the moment of inertia about the pivot point is $I = ml^2$.

$$I\ddot{\theta} = T_c - mgl\sin\theta \quad \rightarrow \quad ml^2\ddot{\theta} + mgl\sin\theta = T_c \quad \rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin\theta = \frac{1}{ml^2}T_c$$

This equation is “nonlinear” due to $\sin\theta$ term.
(1) For small angle variation, since $\sin\theta \approx \theta$, we have linear equation:

$$\ddot{\theta} + \frac{g}{l}\theta = \frac{1}{ml^2}T_c \quad \rightarrow \quad \frac{\Theta(s)}{T_c(s)} = \frac{\frac{1}{ml^2}}{s^2 + \omega_n^2}$$

where $\omega_n = \sqrt{\frac{g}{l}}$ implies the natural frequency of harmonic oscillator.
(2) For general case, we should use MATLAB direct coding or SIMULINK block diagram to simulate nonlinear equation. $\rightarrow$ When $T_c = 1[N/m]$ and $T_c = 4[N/m]$ are applied to the given system, respectively, we can see an increased amplitude and slower frequency becasuse $\sin\theta$ compared to θ signifies a reduced gravational restoring force at the higher angles.