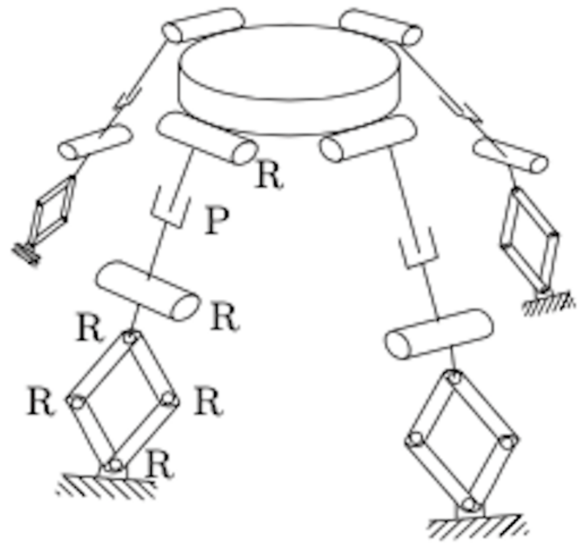
(Question on Chapter 2) Determine DoF of the following spatial parallel mechanism?



(d)

3 Rigid-Body Motions and Twists

- Representations for rigid-body configurations and velocities are derived, similary to rotations and angular velocities.
- Homogeneous transformation matrix $T \in \Re^{4\times4}$ is analogous to the rotation matrix $R \in \Re^{3\times3}$
- Screw axis $\mathcal{S} \in \Re^6$ is analogous to a rotation axis $\hat{\omega} \in \Re^3$
- Twist $\mathcal{V} = \mathcal{S}\dot{\theta} \in \Re^6$ is analogous to an angular velocity $\hat{\omega}\dot{\theta} \in \Re^3$
- Exponential coordinates $\mathcal{S}\theta \in \Re^6$ for rigid-body motions are analogous to exponential coordinates $\hat{\omega}\theta \in \Re^3$ for rotations.

3.1 Homogeneous Transformation Matrices

Consider representations for the combined orientation and position of a rigid body.

- A natural choice would be to use a rotation matrix $R \in SO(3)$ to represent the orientation of the body frame {b} in the fixed frame {s} and a vector $p \in \Re^3$ to represent the origin of {b} in {s}.
- Rather than identifying R and p separately, we package them into a single matrix T as follows.
- Single matrix T will sometimes be denoted (R, p).

Definition 3.4. *The special Euclidean group $SE(3)$, also known as the group of rigid-body motions or homogeneous transformation matrices in $\Re^3$, is the set of all 4×4 real matrices T of the form*

$$T = \begin{bmatrix} R & p \\ 0_{3\times1} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $R \in SO(3)$ and $p \in \Re^3$ is a column vector.

Definition 3.5. *The special Euclidean group $SE(2)$ is in the set of all 3×3 real matrices T of the form*

$$T = \begin{bmatrix} R & p \\ 0_{2\times1} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_1 \\ \sin\theta & \cos\theta & p_2 \\ 0 & 0 & 1 \end{bmatrix}$$

where $R \in SO(2)$, $p \in \Re^2$ is a column vector, and $\theta \in [0, 2\pi)$.

Properties of Transformation Matrices

The following three properties confirm that $SE(3)$ is a group.

Proposition 3.8. *The inverse of a transformation matrix $T \in SE(3)$ is also a transformation matrix, and it has the following form:*

$$T^{-1} = \begin{bmatrix} R & p \\ 0_{3\times1} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0_{3\times1} & 1 \end{bmatrix}$$

Proposition 3.9. *The product of two transformation matrices is also a transformation matrix.*

Proposition 3.10. *The multiplication of transformation matrices is associative, so that $(T_1T_2)T_3 = T_1(T_2T_3)$, but generally not commutative: $T_1T_2 \neq T_2T_1$.*

If '1' is appended to $x \in \Re^3$, making it a four-dimensional vector, the following computation can be performed as a single matrix multiplication:

$$T \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}$$

where the vector $[x^T, 1]^T$ is the representation of x in "homogeneous coordinates", and accordingly $T \in SE(3)$ is called a homogenous transformation. When, by an abuse of notation, we write Tx, we mean $Rx + p$.

Proposition 3.11. *Given $T = (R, p) \in SE(3)$ and $x, y \in \Re^3$, the following hold:*

1. $\|Tx - Ty\| = \|x - y\|$, *where* $\|x\| = \sqrt{x^T x}$.
2. $< Tx - Tz, Ty - Tz > = < x - z, y - z >$, *where* $< x, y > = x^T y$.

- T is regarded as a transformation on points in $\Re^3$
- T transforms a point x to Tx.
- T preserves distances, while T preserves angles.
- If $x, y, z \in \Re^3$ represent the three vertices of a triangle, then the triangle formed by the transformed vertices $\{T_x, T_y, T_z\}$ has the same set of lengths and angles as those of the triangle $\{x, y, z\}$ (the two triangles are said to be isometric).
- Taking $\{x, y, z\}$ to be the points on a rigid body, $\{T_x, T_y, T_z\}$ represents a displaced version of the rigid body.
- $SE(3)$ can be identified with rigid-body motions.

Uses of Transformation Matrices

As was the case for rotation matrices, there are three major uses for a transformation matrix T :

1. to represent the configuration (position and orientation) of a rigid body. (representation)
2. to change the reference frame in which a vector or frame is represented. (operator)
3. to displace a vector or frame. (operator)

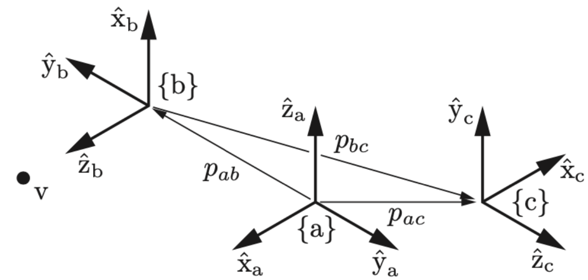


Figure 3.14: Three reference frames in space, and a point v that can be represented in {b} as $v_b = (0, 0, 1.5)$.

Representing a configuration

Let us consider the fixed frame {s} is coincident with {a} and the frames {a}, {b}, and {c}, represented by $T_{sa} = (R_{sa}, p_{sa})$, $T_{sb} = (R_{sb}, p_{sb})$ and $T_{sc} = (R_{sc}, p_{sc})$, respectively, and the locations of the origin of each frame relative to {s} can be written

$$R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad p_{sa} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \quad R_{sc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad p_{sc} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Any frame can be expressed relative to any other frame, for example, $T_{bc} = (R_{bc}, p_{bc})$ represents {b} relative to {c}

$$R_{bc} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \quad p_{bc} = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}$$

It can also be shown using previous Proposition that $T_{cb} = T_{bc}^{-1}$ for any two frames {b} and {c}.

Changing the reference frame of a vector or a frame

By a subscript cancellation rule analogous to that for rotations, for any three reference frames {a}, {b}, and {c}, and any vector v expressed in {b} as v_b,

$$T_{ab}T_{bc} = T_{a\not{b}}T_{\not{b}c} = T_{ac}.$$

$$T_{ab}v_b = T_{a\not{b}}v_{\not{b}} = v_a$$

where v_a is the vector v expressed in {a}.

Displacing (rotating and translating) a vector or a frame

- A transformation matrix T , viewed as the pair $(R, p) = (Rot(\hat{\omega}, \theta), p)$, can act on a frame T_{sb} by rotating it by θ about an axis $\hat{\omega}$ and translating it by p.
- Let us extend the 3×3 rotation operator $R = Rot(\hat{\omega}, \theta)$ to a 4×4 transformations matrices that rotates without translating and translates without rotating, respectively

$$Rotat(\hat{\omega}, \theta) = \begin{bmatrix} R & 0_{1\times3} \\ 0_{3\times1} & 1 \end{bmatrix} \qquad Trans(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The fixed-frame transformation (corresponding to pre-multiplication by $T(R(\hat{\omega}, \theta), p)$) can be interpreted as first rotating the {b} frame by θ about an axis $\hat{\omega}$ in the {s}, then translating it by p in the {s}

$$\begin{aligned} T_{sb'} &= TT_{sb} = Transl(p)Rotat(\hat{\omega}, \theta)T_{sb} \qquad \text{fixed frame} \\ &= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR_{sb} & Rp_{sb} + p \\ 0 & 1 \end{bmatrix} \end{aligned}$$

- The body-frame transformation (corresponding to post-multiplication by $T(R(\hat{\omega}, \theta), p)$) can be interpreted as first translating the {b} frame by p considered to be in the {b} frame, then rotating about $\hat{\omega}$ in the the new body frame.

$$\begin{aligned} T_{sb''} &= T_{sb}T = T_{sb}Transl(p)Rotat(\hat{\omega}, \theta) \qquad \text{body frame} \\ &= \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{sb}R & R_{sb}p + p_{sb} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

- (In the previous lecture) Pre-multiplying by $R = Rot(\hat{\omega}, \theta)$ yields a rotation about an axis $\hat{\omega}$ considered to be in the fixed frame, and post-multiplying by R yields a rotation about $\hat{\omega}$ considered as being in the body frame.

$$R_{sb'} = \text{rotate by } R \text{ in \{s\} frame } (R_{sb}) = RR_{sb}$$

$$R_{sb''} = \text{rotate by } R \text{ in \{b\} frame } (R_{sb}) = R_{sb}R$$

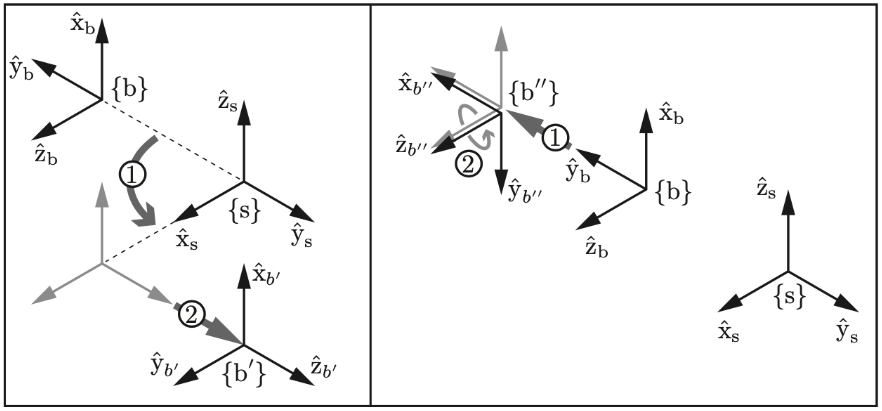


Figure 3.15: Fixed-frame and body-frame transformations corresponding to $\hat{\omega} = (0,0,1)$, $\theta = 90°$, and $p = (0,2,0)$. (Left) The frame {b} is rotated by 90° about $\hat{z}_s$ and then translated by two units in $\hat{y}_s$, resulting in the new frame {b′}. (Right) The frame {b} is translated by two units in $\hat{y}_b$ and then rotated by 90° about its $\hat{z}$ axis, resulting in the new frame {b″}.

$$T = T(Rot(\hat{\omega},\theta),p) = Transl(p)Rotat(\hat{\omega},\theta) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{sb} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

New frame {b'} achieved by a fixed-frame transformation TT_{sb} and the new frame {b"} achieved by a body-frame transformation $T_{sb}T$ are given by

$$TT_{sb} = Transl(p)Rotat(\hat{\omega},\theta)T_{sb} = T_{sb'} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{sb}T = T_{sb}Transl(p)Rotat(\hat{\omega},\theta) = T_{sb''} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

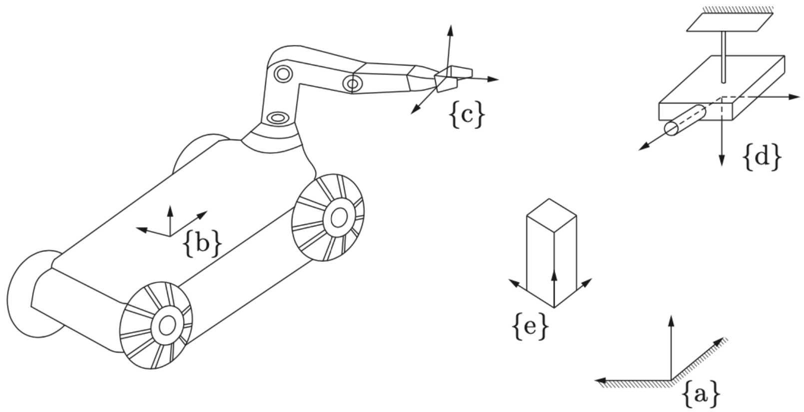


Figure 3.16: Assignment of reference frames.

Example 3.2. *A robot arm mounted on a wheeled mobile platform moving in a room, and a camera fixed to the ceiling. Find T_{ce} ? (in order to calculate how to move the robot arm so as to pick up the object, the configuration of the object relative to the robot hand)*

- *Frame $\{b\}$ is attached to the wheeled platform*
- *Frame $\{c\}$ is attached to the end-effector of the robot arm*
- *Frame $\{d\}$ is attached to the camera.*
- *A fixed frame $\{a\}$ is established and the robot must pick up an object with body frame $\{e\}$*
- *The transformations T_{db} and T_{de} can be calculated from measurements obtained with the camera.*
- *The transformation T_{bc} can be calculated using the arm's joint-angle measurements.*
- *The transformation T_{ad} is assumed to be known in advance.*

$$T_{ab}T_{bc}T_{ce} = T_{ad}T_{de} \quad \rightarrow \quad T_{ce} = (T_{ab}T_{bc})^{-1}T_{ad}T_{de} = (T_{ad}T_{db}T_{bc})^{-1}T_{ad}T_{de}$$