2.2 Angular Velocities

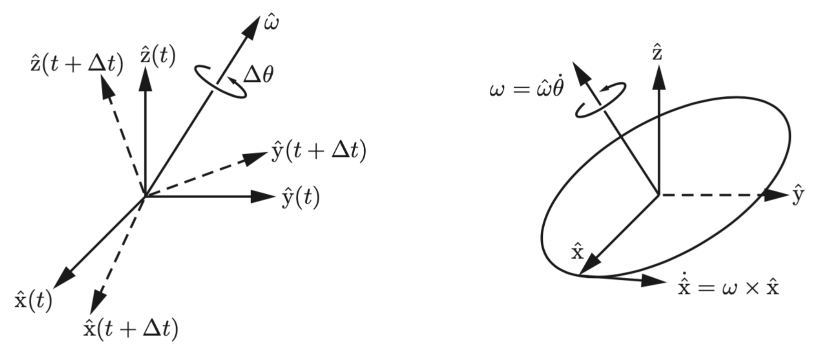

Figure 3.10: (Left) The instantaneous angular velocity vector. (Right) Calculating $\dot{\hat{x}}$.

- Suppose that a frame with unit axes $\{\hat{x}, \hat{y}, \hat{z}\}$ is attached to a rotating body. For given the rate of rotation $\dot{\theta}$ and the instantaneous axis of rotation $\hat{\mathrm{w}}$, the angular velocity w is defined as follows:

$$\mathrm{w} = \hat{\mathrm{w}}\dot{\theta}$$

- Let us determine the time derivatives of these unit axes

$$\dot{\hat{x}} = \mathrm{w} \times \hat{x} \qquad \dot{\hat{y}} = \mathrm{w} \times \hat{y} \qquad \dot{\hat{z}} = \mathrm{w} \times \hat{z}$$

- Let $R(t)$ be the rotation matrix describing the orientation of the body frame w.r.t. the fixed frame at time t, and thus we have $R(t) = [\hat{x}, \hat{y}, \hat{z}] = [r_1, r_2, r_3]$ in the fixed-frame coordinates.
- At a specific time t, let $\omega_s \in \Re^3$ be the angular velocity w expressed in fixed-frame coordinates. Above equations can be expressed in fixed-frame coordinates as

$$\dot{r}_i = \omega_s \times r_i \quad \textbf{for} \quad i = 1, 2, 3 \qquad \rightarrow \qquad \dot{R} = \omega_s \times R$$

Skew-symmetric matrix representation

- To eliminate the cross product, let us introduce new notation $[\omega_s]$ as 3×3 skew-symmetric matrix representation of $\omega_s \in \Re^3$. Then we have

$$\dot{R} = \omega_s \times R = [\omega_s]R$$

Definition 3.3. *Given a vector* $x = [x_1, x_2, x_3]^T \in \Re^3$*, define*

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

The matrix $[x]$ *is a* 3×3 *skew-symmetric matrix representation of* x*; that is,*

$$[x] = -[x]^T.$$

The set of all 3×3 *real skew-symmetric matrices is called* $so(3)$*.*

Proposition 3.5. *Given any* $\omega \in \Re^3$ *and* $R \in SO(3)$*, the following always holds:*

$$R[\omega]R^T = [R\omega]$$

- With the skew-symmetric notation, we can get the following equation:

$$[\omega_s]R = \dot{R} \quad \rightarrow \quad [\omega_s] = \dot{R}R^{-1}$$

- Now let ω_b be w expressed in body-frame coordinates. To see how to obtain ω_b from ω_s and vice versa, we write R explicitly as R_{sb}. By our subscript cancellation rule, $\omega_s = R_{sb}\omega_b$, we have

$$\omega_b = R_{sb}^{-1}\omega_s = R^{-1}\omega_s = R^T\omega_s$$

- Let us now express this relation in skew-symmetric matrix form:

$$[\omega_b] = [R^T\omega_s] = R^T[\omega_s]R = R^T\dot{R}R^TR = R^T\dot{R} = R^{-1}\dot{R}$$

Proposition 3.6. *Let $R(t) = R_{sb}$ denote the orientation of the rotating frame as seen from the fixed frame. Denote by* w *the angular velocity of the rotating frame. Then*

$$\dot{R}R^{-1} = [\omega_s] \qquad\qquad R^{-1}\dot{R} = [\omega_b]$$

- $\omega_s \in \Re^3$ is the fixed-frame vector representation of w and $[\omega_s] \in so(3)$ is its 3×3 matrix representation. Note that ω_s is independent of the choice of body frame, although it may appear to depend on both frames from $\dot{R}R^{-1}$.

- $\omega_b \in \Re^3$ is the body-frame vector representation of w, and ω_b is independent of the choice of fixed frame.

2.3 Exponential Coordinate Representation of Rotation

- The exponential coordinates parametrize a rotation matrix in terms of a rotation axis (represented by a unit vector $\hat{\omega}$) and an angle of rotation θ about that axis;

$$\hat{\omega}\theta \in \Re^3$$

  where it is called axis-angle representation of a rotation

- The exponential coordinate representation $\hat{\omega}\theta$ for a rotation matrix R can be interpreted equivalently as:
  - the axis $\hat{\omega}$ and rotation angle θ such that, if a frame initially coincident with {s} were rotated by θ about $\hat{\omega}$, its final orientation relative to {s} would be expressed by R.
  - the angular velocity $\hat{\omega}\theta$ expressed in {s} such that, if a frame initially coincident with {s} followed $\hat{\omega}\theta$ for one unit of time, its final orientation would be expressed by R
  - the angular velocity $\hat{\omega}$ expressed in {s} such that, if a frame initially coincident with {s} followed for θ units of time, its final orientation would be expressed by R.

- Latter two views suggest that we consider exponential coordiantes in the setting of linear differential equations.

Essential Results from Linear Differential Equations Theory

- Let us begin with the simple scalar linear differential equation using the initial condition $x_0 = x(0) \in \Re$ from time 0 to t

$$\dot{x}(t) = ax(t) \quad \rightarrow \quad \frac{dx}{dt} = ax \quad \rightarrow \quad \frac{dx}{x} = adt \quad \rightarrow$$
$$\ln x|_{x(0)}^{x(t)} = a(t-0) \quad \rightarrow \quad \ln\frac{x(t)}{x(0)} = at \quad \rightarrow \quad \therefore \quad x(t) = e^{at}x_0$$

  where series expansion of exponential function is

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

- Now consider the vector linear differential equation with a n-dimensional $x_0 \in \Re^n$

$$\dot{x}(t) = Ax(t) \quad \rightarrow \quad x(t) = e^{At}x_0$$

  where $A \in \Re^{n\times n}$ and its matrix exponential $e^{At} \in \Re^{n\times n}$ is defined as

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

  in which the convergence and existence of the matrix exponential are guaranted, but we will skip the proofs.

- While $AB \neq BA$ for arbitrary square matrices A and B, it is always true that

$$Ae^{At} = e^{At}A$$

- How to obtain the matrix exponential as a closed-form: using the diagonalization technique $A = PDP^{-1}$

$$
\begin{aligned}
e^{At} &= I + (PDP^{-1})t + (PDP^{-1})(PDP^{-1})\frac{(t)^2}{2!} + (PDP^{-1})(PDP^{-1})(PDP^{-1})\frac{(t)^3}{3!} + \cdots \\
&= I + (PDP^{-1})t + (PD^2P^{-1})\frac{(t)^2}{2!} + (PD^3P^{-1})\frac{(t)^3}{3!} + \cdots \\
&= P\left(I + Dt + \frac{(Dt)^2}{2!} + \frac{(Dt)^3}{3!} + \cdots\right)P^{-1} \\
&= Pe^{Dt}P^{-1}
\end{aligned}
$$

- Since D is diagonal, i.e., $D = diag(d_1, d_2, \cdots, d_n)$, then its matrix exponential is particularly simple to evaluate

$$
e^{Dt} = \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_2 t} & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & e^{d_n t} \end{bmatrix} \in \Re^{n \times n}
$$

- Please refer to Proposition 3.10 in the textbook!

Exponential Coordinates of Rotations

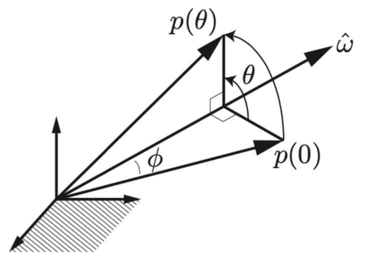

Figure 3.11: The vector $p(0)$ is rotated by an angle θ about the axis $\hat{\omega}$, to $p(\theta)$.

- Suppose that a three-dimensional $p(0) \in \Re^3$ is rotated by θ about $\hat{\omega}$ to $p(\theta)$; where we assume that all quantities are expressed in fixed-frame coordinates.
- This rotation can be achieved by imagining that $p(0)$ rotates at a constant rate of 1rad/s from time $t = 0$ to $t = \theta$.
- Let $p(t)$ denote the path traced by the tip of the vector. The velocity of $p(t)$, denoted $\dot{p}$, is then given by

$$\dot{p} = \hat{\omega} \times p = [\hat{\omega}]p \quad \rightarrow \quad p(t) = e^{[\hat{\omega}]t}p(0) \quad \rightarrow \quad \therefore \quad p(\theta) = e^{[\hat{\omega}]\theta}p(0)$$

- Since $[\hat{\omega}]^3 = -[\hat{\omega}]$, $[\hat{\omega}]^4 = -[\hat{\omega}]^2$, and $[\hat{\omega}]^5 = [\hat{\omega}]$, the matrix exponential $e^{[\hat{\omega}]\theta}$ in series form is

$$\begin{aligned} e^{[\hat{\omega}]\theta} &= I + [\hat{\omega}]\theta + [\hat{\omega}]^2\frac{\theta^2}{2!} + [\hat{\omega}]^3\frac{\theta^3}{3!} + [\hat{\omega}]^4\frac{\theta^4}{4!} + [\hat{\omega}]^5\frac{\theta^5}{5!} + \cdots \\ &= I + [\hat{\omega}]\theta + [\hat{\omega}]^2\frac{\theta^2}{2!} - [\hat{\omega}]\frac{\theta^3}{3!} - [\hat{\omega}]^2\frac{\theta^4}{4!} + [\hat{\omega}]\frac{\theta^5}{5!} + \cdots \\ &= I + \left(\theta - \frac{\theta^3}{3!} + \cdots\right)[\hat{\omega}] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right)[\hat{\omega}]^2 = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \end{aligned}$$

because the series expansions for $\sin\theta$ and $\cos\theta$:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

Proposition 3.7. *(Rodrigues' formula for rotation) Given a vector $\hat{\omega}\theta \in \Re^3$ such that θ is any scalar and $\hat{\omega} \in \Re^3$ is a unit vector, the matrix exponential of $[\hat{\omega}] \in so(3)$ is*

$$\begin{aligned}
Rot(\hat{\omega},\theta) &= e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} + (1-\cos\theta) \begin{bmatrix} -(\hat{\omega}_2^2+\hat{\omega}_3^2) & \hat{\omega}_1\hat{\omega}_2 & \hat{\omega}_1\hat{\omega}_3 \\ \hat{\omega}_1\hat{\omega}_2 & -(\hat{\omega}_1^2+\hat{\omega}_3^2) & \hat{\omega}_2\hat{\omega}_3 \\ \hat{\omega}_1\hat{\omega}_3 & \hat{\omega}_2\hat{\omega}_3 & -(\hat{\omega}_1^2+\hat{\omega}_2^2) \end{bmatrix} \\
&= \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}
\end{aligned}$$

note that $\hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2 = 1$, $c_\theta = \cos\theta$, and $s_\theta = \sin\theta$.

- Also

$R' = e^{[\hat{\omega}]\theta}R = Rot(\hat{\omega},\theta)R$ orientation achived by rotating R by θ about the axis $\hat{\omega}$ in the fixed frame

$R'' = Re^{[\hat{\omega}]\theta} = RRot(\hat{\omega},\theta)$ orientation achived by rotating R by θ about the axis $\hat{\omega}$ in the body frame

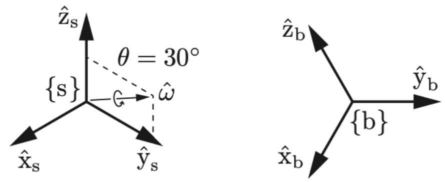

Figure 3.12: The frame {b} is obtained by a rotation from {s} by $\theta_1 = 30°$ about $\hat{\omega}_1 = (0, 0.866, 0.5)$.

Example 3.1. *The frame {b} in Figure 3.12 is obtained by rotation from an initial orientation aligned with the fixed frame {s} about a unit axis* $\hat{\omega} = (0, 0.866, 0.5)$ *by an angle* $\theta = 30° = 0.524 rad$*. Since* $s_\theta = \sin\theta = 0.5$ *and* $c_\theta = \cos\theta = 0.866$*, we have*

$$R = e^{[\hat{\omega}]\theta} = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix} = \begin{bmatrix} 0.866 & -0.250 & 0.433 \\ 0.250 & 0.967 & 0.058 \\ -0.433 & 0.058 & 0.899 \end{bmatrix}$$

Exponetial coordinates and matrix logarithm of rotation R *are, respectively,*

$$\hat{\omega}\theta = \begin{bmatrix} 0 \\ 0.453 \\ 0.262 \end{bmatrix} \quad \textit{and} \quad [\omega]\theta = [\omega\theta] = \begin{bmatrix} 0 & -0.262 & 0.453 \\ 0.262 & 0 & 0 \\ -0.453 & 0 & 0 \end{bmatrix}$$

If $\hat{\omega}\theta \in \Re^3$ *represents the exponential coordinates of a rotation matrix* R*, then the skew-symmetric matrix* $[\omega]\theta = [\omega\theta] \in \Re^{3\times3}$ *is the matrix logarithm of a rotation* R*.*

Matrix Logarithm of Rotations

- From the exponential coordinates $\hat{\omega}\theta$,

$$\begin{aligned} \text{matrix exponetial :} \quad & [\hat{\omega}]\theta \in so(3) \quad \rightarrow \quad R = e^{[\hat{\omega}]\theta} \in SO(3) \\ \text{matrix logarithm :} \quad & R = e^{[\hat{\omega}]\theta} \in SO(3) \quad \rightarrow \quad [\hat{\omega}]\theta \in so(3) \end{aligned}$$

- Let us derive the matrix logarithm from $R = e^{[\hat{\omega}]\theta}$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

- Subtracting the transpose from both sides leads to the following

$$R - R^T = \begin{bmatrix} 0 & r_{12} - r_{21} & r_{13} - r_{31} \\ r_{21} - r_{12} & 0 & r_{23} - r_{32} \\ r_{31} - r_{13} & r_{32} - r_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2\hat{\omega}_3 s_\theta & 2\hat{\omega}_2 s_\theta \\ 2\hat{\omega}_3 s_\theta & 0 & -2\hat{\omega}_1 s_\theta \\ -2\hat{\omega}_2 s_\theta & 2\hat{\omega}_1 s_\theta & 0 \end{bmatrix}$$

- If $\sin\theta \neq 0$, then we can get the skew-symmetric matrix form of the rotation axis $\hat{\omega}$ by divding $2\sin\theta$ and take the trace

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^T) = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix}$$

- For the rotation angle θ about the rotation axix $\hat{\omega}$ from R, let us take the trace

$$\begin{aligned} tr(R) &= r_{11} + r_{22} + r_{33} = 3c_\theta + (\hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2)(1 - c_\theta) \\ &= 1 + 2\cos\theta \qquad \rightarrow \qquad \theta = \cos^{-1}\left(\frac{tr(R) - 1}{2}\right) \end{aligned}$$

  note that $\hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2 = 1$.

- Recall that $\hat{\omega}$ represents the axis of rotation for the given R. Because of the $\sin\theta$ term in the denominator, $[\hat{\omega}]$ is not well defined if θ is an integer multiple of π.

- Let us now return to the case $\theta = k\pi$, where k is some integer.

  - When k is an even integer, regardless of $\hat{\omega}$ we have rotated back to $R = I$ so the vector $\hat{\omega}$ is undefined.
  - When k is an odd integer (corresponding to $\theta = \pm\pi, \pm 3\pi, \cdots$ which in turn implies $tr(R) = -1$), the exponential formula simplifies to

$$R = e^{[\hat{\omega}]\theta} = I + 2[\hat{\omega}]^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} -(\hat{\omega}_2^2 + \hat{\omega}_3^2) & \hat{\omega}_1\hat{\omega}_2 & \hat{\omega}_1\hat{\omega}_3 \\ \hat{\omega}_1\hat{\omega}_2 & -(\hat{\omega}_1^2 + \hat{\omega}_3^2) & \hat{\omega}_2\hat{\omega}_3 \\ \hat{\omega}_1\hat{\omega}_3 & \hat{\omega}_2\hat{\omega}_3 & -(\hat{\omega}_1^2 + \hat{\omega}_2^2) \end{bmatrix}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 - 2(\hat{\omega}_2^2 + \hat{\omega}_3^2) & 2\hat{\omega}_1\hat{\omega}_2 & 2\hat{\omega}_1\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_2 & 1 - 2(\hat{\omega}_1^2 + \hat{\omega}_3^2) & 2\hat{\omega}_2\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_3 & 2\hat{\omega}_2\hat{\omega}_3 & 1 - 2(\hat{\omega}_1^2 + \hat{\omega}_2^2) \end{bmatrix}$$

- Three diagonal terms can be manipulated as

$$r_{ii} = 1 - 2(\hat{\omega}_j^2 + \hat{\omega}_k^2) = 1 - 2(1 - \hat{\omega}_i^2) \qquad \rightarrow \qquad \hat{\omega}_i = \sqrt{\frac{r_{ii}+1}{2}}$$

  from $\hat{\omega}_i^2 + \hat{\omega}_j^2 + \hat{\omega}_k^2 = 1$.
- Off-diagonal terms lead to the following three equations:

$$2\hat{\omega}_i\hat{\omega}_j = r_{ij} = r_{ji}$$

- For example, if $tr(R) = -1$ then $\theta = \pi$, and the axis of rotation is described by

$$\begin{aligned} \hat{\omega}_1 &= \sqrt{\frac{r_{11}+1}{2}} \\ \hat{\omega}_2 &= \frac{r_{21}}{\sqrt{2(r_{11}+1)}} \qquad \leftarrow \quad 2\hat{\omega}_1\hat{\omega}_2 = r_{21} \\ \hat{\omega}_3 &= \frac{r_{31}}{\sqrt{2(r_{11}+1)}} \qquad \leftarrow \quad 2\hat{\omega}_1\hat{\omega}_3 = r_{31} \end{aligned}$$

  Note that $r_{12} = r_{21}$, $r_{13} = r_{31}$, and $r_{23} = r_{32}$ when $\theta = \pi$.

Algorithm 3.1. *Given* $R \in SO(3)$*, find* $\theta \in [0, \pi]$ *and a unit rotation axis* $\hat{\omega} \in \Re^3$*,* $\|\hat{\omega}\| = 1$ *such that* $e^{[\hat{\omega}]\theta} = R$*. The vector* $\hat{\omega}\theta \in \Re^3$ *comprises the exponential coordinates for* R *and skew-symmetric matrix* $[\hat{\omega}]\theta \in so(3)$ *is the matrix logarithm of* R*.*

- *If* $R = I$*, then* $\theta = 0$ *and* $\hat{\omega}$ *is undefined*
- *If* $tr(R) = -1$*, then* $\theta = \pi$*. Set* $\hat{\omega}$ *equal to any of the following three vectors that is a feasible solution:*

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1 + r_{33} \end{bmatrix} = \frac{1}{\sqrt{2(1+r_{22})}} \begin{bmatrix} r_{12} \\ 1 + r_{22} \\ r_{32} \end{bmatrix} = \frac{1}{\sqrt{2(1+r_{11})}} \begin{bmatrix} 1 + r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

  Note that if $\hat{\omega}$ *is a solution, then so is* $-\hat{\omega}$*.*

- *Otherwise* $\theta = \cos^{-1}\left(\frac{tr(R)-1}{2}\right) \in [0, \pi)$ *and*

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^T)$$

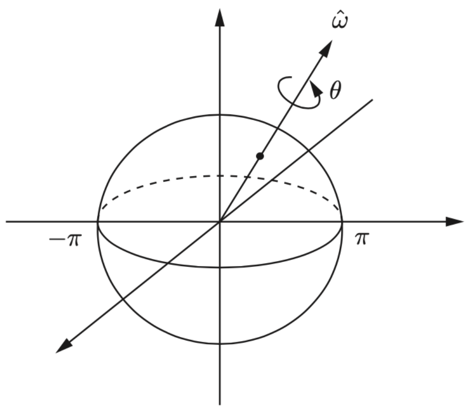

Figure 3.13: $SO(3)$ as a solid ball of radius π. The exponential coordinates $r = \hat{\omega}\theta$ may lie anywhere within the solid ball.

- Because the matrix logarithm calculates exponential coordinates $\hat{\omega}\theta$ satisfying $\|\hat{\omega}\theta\| \leq \pi$, we can picture the rotation group $SO(3)$ as a solid ball of radius π
- Given a point $r \in \Re^3$ in this solid ball, let $\hat{\omega} = \frac{r}{\|r\|}$ be the unit axis in the direction from the origin to the point r and let $\theta = \|r\|$ be the distance from the origin to r, so that $r = \hat{\omega}\theta$.
- For any $R \in SO(3)$ such that $tr(R) \neq -1$, there exists a unique r in the interior of the solid ball such that $e^{[r]} = R$.
- In the event that $tr(R) = -1$, $\log R$ is given by two antipodal points on the surface of this solid ball. That is, if there exists some r such that $R = e^{[r]}$ with $\|r\| = \pi$ then $R = e^{[-r]}$ also holds; both r and $-r$ correspond to the same rotation R.