(NC) 1.1 Nonlinear Model

1. Dynamical systems?

2. (Dynamical systems) are modeled by a finite number of coupled 1st-order ordinary differential equations (ODE):

$$
\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \cdots, x_n, u_1, \cdots, u_m) \\
\dot{x}_2 &= f_2(t, x_1, \cdots, x_n, u_1, \cdots, u_m) \\
\vdots \quad &\quad \vdots \\
\dot{x}_n &= f_n(t, x_1, \cdots, x_n, u_1, \cdots, u_m)
\end{aligned}
$$

   where

$$
\begin{aligned}
\dot{x}_i &: \text{the derivative of } x_i \text{ w.r.t. the time variable } t \\
u_1, \cdots, u_m &: \text{the input variables} \\
x_1, \cdots, x_n &: \text{the state variables representing the memory that the dynamical system has of its past}
\end{aligned}
$$

3. (Vector Notation) It can be written as one n-dimensional 1st-order vector differential equation:

$$\dot{x} = f(u, x, u) \tag{1}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \Re^n \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \in \Re^m \qquad f(t, x, u) = \begin{bmatrix} f_1(t, x, u) \\ f_2(t, x, u) \\ \vdots \\ f_n(t, x, u) \end{bmatrix} \in \Re^n$$

4. (Output Equation) Variables measured physically by sensors are expressed from the dynamical system:

$$y = h(t, x, u) \in \Re^q \tag{2}$$

5. (Unforced State Equation) If the input $u = \gamma(t, x)$ is designed and applied to the state equation, *e.g.,* $\dot{x} = f(t, x, \gamma(t, x))$, then it yields the unforced state equation:

$$\dot{x} = f(t, x) \tag{3}$$

6. (Piecewise Continuous in t and Locally Lipschitz in x) In dealing with $\dot{x} = f(t,x)$, we require the function $f(t,x)$ to be piecewise continuous in t and locally Lipschitz in x over the domain of interest.

   (1) For a fixed x,

   the function $f(t,x)$ is piecewise continuous in t on an interval $J \subset \Re$,
   if f is continuous in t for all $t \in J$ except a finite number of points where f may have finite-jump discontinuities.
   $\Rightarrow$ the input $u(t)$ can be designed to have step changes with time.

   (2) For all $t \in J$,

   a function $f(t,x)$ is locally Lipschitz in x at a point x_0,
   if there is a neighborhood $N(x_0, r) = \{\|x - x_0\| < r\}$ and a positive constant L such that $f(t,x)$ satisfies the Lipschitz condition:

$$\|f(t,x) - f(t,y)\| \le L\|x - y\| \tag{4}$$

   for all $x, y \in N(x_0, r)$, where

$$\|x\| = \sqrt{x^T x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{5}$$

   $\Rightarrow$ A function $f(x)$ that has infinite slope at one point is not locally Lipschitz at that point.
   $\Rightarrow$ For $t \in J \subset \Re$ and $x \in D \subset \Re^n$, if the function $f(t,x)$ and its partial derivatives $\frac{\partial f_i}{\partial x_j}$ are continuous and bounded, then the function $f(t,x)$ is locally Lipschitz in x on D.
   $\Rightarrow$ For $t \in \Re$ and $x \in \Re^n$, if the function $f(t,x)$ and its partial derivatives $\frac{\partial f_i}{\partial x_j}$ are continuous and globally bounded, then the function $f(t,x)$ is globally Lipschitz in x on $\Re^n$.

7. (Example 1.1) Find the Lipschitz constant L over the set $W = \{|x_1| \leq a, |x_2| \leq a\}$?

$$f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$$

Notice that $|a+b| \leq |a| + |b|$ and $|a||b| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$

$$\begin{aligned}
|f_1(x) - f_1(y)| &= |-x_1 + x_1x_2 - (-y_1 + y_1y_2)| = |(y_1 - x_1) + (x_1x_2 - y_1y_2)| \\
&\leq |y_1 - x_1| + |x_1x_2 - y_1y_2| = |y_1 - x_1| + |x_1x_2 - x_1y_2 + x_1y_2 - y_1y_2| \\
&= |y_1 - x_1| + |x_1(x_2 - y_2) + y_2(x_1 - y_1)| \\
&\leq |y_1 - x_1| + a|x_2 - y_2| + a|x_1 - y_1| = (a+1)|x_1 - y_1| + a|x_2 - y_2| \\
|f_2(x) - f_2(y)| &\leq a|x_1 - y_1| + (a+1)|x_2 - y_2|
\end{aligned}$$

$$\begin{aligned}
\|f(x) - f(y)\|^2 &= |f_1(x) - f_1(y)|^2 + |f_2(x) - f_2(y)|^2 \\
&= (a+1)^2|x_1 - y_1|^2 + a^2|x_2 - y_2|^2 + 2a(a+1)|x_1 - y_1||x_2 - y_2| \\
&\quad + a^2|x_1 - y_1|^2 + (a+1)^2|x_2 - y_2|^2 + 2a(a+1)|x_1 - y_1||x_2 - y_2| \\
&= (2a^2 + 2a + 1)|x_1 - y_1|^2 + (2a^2 + 2a + 1)|x_2 - y_2|^2 + 4a(a+1)|x_1 - y_1||x_2 - y_2| \\
&\leq (2a^2 + 2a + 1)|x_1 - y_1|^2 + (2a^2 + 2a + 1)|x_2 - y_2|^2 + 2a(a+1)|x_1 - y_1|^2 + 2a(a+1)|x_2 - y_2|^2 \\
&= (4a^2 + 4a + 1)|x_1 - y_1|^2 + (4a^2 + 4a + 1)|x_2 - y_2|^2 = (2a+1)^2\left\{|x_1 - y_1|^2 + |x_2 - y_2|^2\right\} \\
&= (2a+1)^2\|x - y\|^2
\end{aligned}$$

Therefore, since $\|f(x) - f(y)\| \leq (2a+1)\|x - y\|$, f is Lipschitz on W with the Lipschitz constant

$$\therefore \quad L = 2a + 1$$

8. Why Lipschitz condition?
$\Rightarrow$ The local Lipschitz property is related to local existence and uniqueness of the solution of the state equation $\dot{x} = f(t, x)$

9. (Lemma 1.3) (Existence and Uniqueness) Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $D \subset \Re^n$. Let W be a compact (closed and bounded) subset of D, $x_0 \in W$, and suppose it is known that every solution of

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \tag{6}$$

lies entirely in W. Then there is a unique solution that is defined for all $t \geq t_0$

10. (Example 1.3/1.4) Check whether it has a unique solution?

$$(a)\ \ \dot{x} = -x^3 \qquad (b)\ \ \dot{x} = -x^2 \qquad (c)\ \ \dot{x} = x^{1/3}$$

(a) 1. $f(x) = -x^3$ is locally Lipschitz b/c $f'(x) = -3x^2$ is continuous and locally bounded for all x in a domain $D \subset \Re$.
2. At any instant of time, if $x(t) > 0$, then $\dot{x}(t) < 0$ and $x(t)$ will be decreasing. Similarly, if $x(t) < 0$, then $\dot{x}(t) > 0$ and $x(t)$ will be increasing. Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $\{|x| \leq |a|\}$.
Thus we conclude by Lemma 1.3 that $\dot{x} = -x^3$ has a unique solution for all $t \geq 0$

(b) 1. $f(x) = -x^2$ is locally Lipschitz b/c $f'(x) = -2x$ is continuous and locally bounded for all x in a domain $D \subset \Re$.
2. At any instant of time, if $x(t) > 0$, then $\dot{x}(t) < 0$ and $x(t)$ will be decreasing. However, if $x(t) < 0$, then $\dot{x}(t) < 0$ and $x(t)$ will be decreasing. Therefore, starting from any initial condition $x(0) = a < 0$, the solution does not stay in any compact set, but goes to infinity in finite time
3. Consider $x(0) = -1$, then the solution does not exist at $t = 1$.

$$\frac{dx}{dt} = -x^2 \quad \rightarrow \quad -\frac{dx}{x^2} = dt \quad \rightarrow \quad -\int_{-1}^{x(t)} x^{-2} dx = \int_0^t dt \quad \rightarrow \quad \frac{1}{x}\Big|_{-1}^{x(t)} = t \quad \rightarrow \quad x(t) = \frac{1}{t-1}$$

(c) 1. $f(x) = x^{1/3}$ is not locally Lipschitz b/c $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$ is not bounded at $x = 0$.
2. Consider $x(0) = 0$, then we have two different solutions $x(t) = 0$ and $x(t) = (2t/3)^{3/2}$

$$\frac{dx}{dt} = x^{1/3} \quad \rightarrow \quad \frac{dx}{x^{1/3}} = dt \quad \rightarrow \quad \int_0^{x(t)} x^{-1/3} dx = \int_0^t dt \quad \rightarrow \quad \frac{3}{2}x^{2/3}\Big|_0^{x(t)} = t \quad \rightarrow \quad x(t) = (2t/3)^{3/2}$$

11. (Autonomous System) A special case of $\dot{x} = f(t, x)$ arises when the function f does not depend explicitly on t; that is

$$\dot{x} = f(x) \tag{7}$$

$\Rightarrow$ It is said to be autonomous system.
$\Rightarrow$ The behavior of an autonomous system is invariant to shifts in the time origin.
$\Rightarrow$ All autonomous systems are time invariant, but the converse is not true.

12. (Time Invariant System) More generally, the state model $\dot{x} = f(t, x, u)$ and $y = h(t, x, u)$ is said to be time invariant if the functions f and h do not depend explicitly on t; that is,

$$\dot{x} = f(x, u) \;\; \text{and} \;\; y = h(x, u) \tag{8}$$

$\Rightarrow$ However it is not called as autonomous system b/c it has an input u.
$\Rightarrow$ If either f or h depends explicitly on t, then the state model is said to be time varying.

13. (Transformation or Mapping) Transforming (or mapping) the state equation from x-coordinate into z-coordinate is often required, $z = T(x)$.

14. (Lemma 1.4) (Diffeomorphism) The continuously differentiable map $z = T(x)$ is a local diffeomorphism at x_0, if the Jacobian matrix $[\frac{\partial T}{\partial x}]$ is nonsingular at x_0. It is a global diffeomorphism if and only if $[\frac{\partial T}{\partial x}]$ is nonsingular for all $x \in \Re^n$ and T is proper; that is, $\lim_{\|x\| \to \infty} \|T(x)\| = \infty$.

15. (Example 1.5) Assume that $h(x_1)$ is continuously differentiable. Consider the map and its Jacobian

$$z = T(x) = \begin{bmatrix} -h(x_1) - x_2/\epsilon \\ x_1 \end{bmatrix} \quad \rightarrow \quad \frac{\partial T}{\partial x} = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -h'(x_1) & -\frac{1}{\epsilon} \\ 1 & 0 \end{bmatrix}$$

(1) Since the determinant of $\frac{\partial T}{\partial x}$ is $\frac{1}{\epsilon} > 0$, the Jacobian matrix is nonsingular for all $x \in \Re^2$

(2) Since

$$\lim_{\|x\| \to \infty} \|T(x)\| = \lim_{\|x\| \to \infty} \sqrt{[h(x_1) + x_2/\epsilon]^2 + x_1^2} = \infty$$

the map $T(x)$ is proper.

(3) Thus the map $z = T(x)$ is a global diffeomorphism

16. (Equilibrium Points) Equilibrium points are important features of the state equation. They are the real solutions of

$$f(t,x) = 0 \quad \textbf{from} \quad \dot{x} = f(t,x) = 0$$

17. (For time invariant systems) the equilibrium points are the real solutions of

$$f(x) = 0 \quad \textbf{from} \quad \dot{x} = f(x) = 0 \tag{9}$$

For example, the pendulum equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin x_1 - bx_2 \end{bmatrix}$$

has equilibrium points at $x_1 = n\pi, x_2 = 0$ for $n = 0, \pm 1, \pm 2, \cdots$

18. (For linear systems) the equilibrium point is either $x = 0$ if A is nonsingular or homogeneous solution of Ax if A is singular.

$$Ax = 0 \quad \textbf{from} \quad \dot{x} = Ax = 0 \tag{10}$$

For example, the linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

has equilibrium points at $x_1 = \alpha \in \Re$ and $x_2 = 0$ for any real constant α.

(NC) 1.2 Nonlinear Phenomena

1. Linear systems?

2. Superposition principle $\Rightarrow$ a powerful tool for linear system analysis

$$\begin{aligned} \dot{x}_1 &= Ax_1 + Bu_1 \\ \dot{x}_2 &= Ax_2 + Bu_2 \\ (\alpha\dot{x}_1 + \beta\dot{x}_2) &= A(\alpha x_1 + \beta x_2) + B(\alpha u_1 + \beta u_2) \end{aligned}$$

3. For nonlinear systems, the superposition principle no longer holds

4. The first step in analyzing a nonlinear system is usually to linearize it about some nominal operating point and to analyze the resulting linear model.

$$\dot{x} = f(x) \quad \rightarrow \quad \dot{\bar{x}} = \left[\left. \frac{\partial f(x)}{\partial x} \right|_{x=x_o} \right] \bar{x} = A\bar{x} \quad \textbf{with} \quad \bar{x} = x - x_o$$

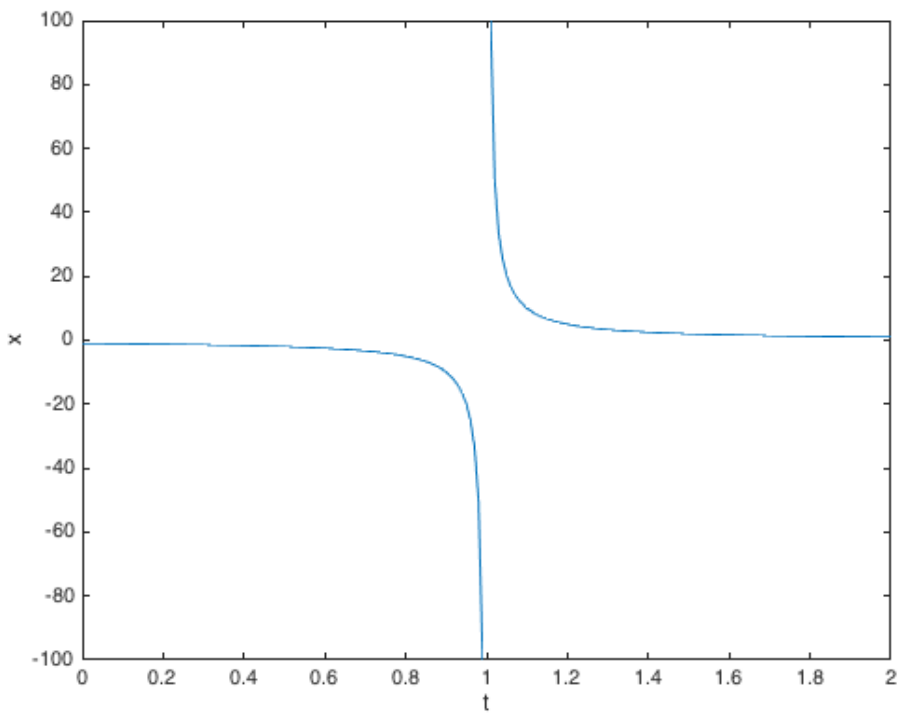
5. Big limitation of linearization $\Rightarrow$ it can only predict the local behavior of the nonlinear system in the vicinity of that point.

6. Nonlinear phenomena that can take place only in the presence of nonlinearity

(Finite escape time)

- The state of an unstable linear system $x \to \infty$ as $t \to \infty$
- The state of a nonlinear system can go to infinity in finite time. (Example 1.3)

$$\dot{x} = -x^2 \text{ with } x(0) = -1 \quad \to \quad x(t) = \frac{1}{t-1}$$



(Multiple isolated equilibria)

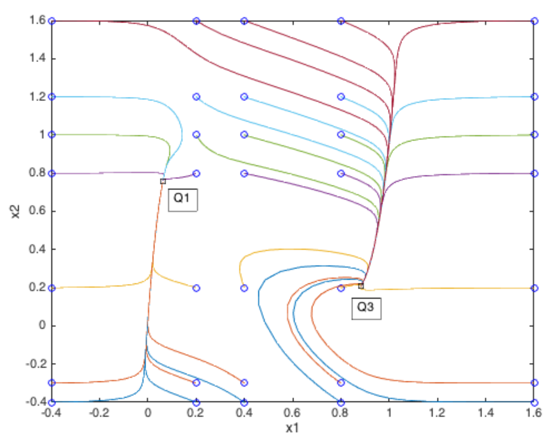
- A linear system can have only one isolated equilibrium point
- A nonlinear system can have more than one isolated equilibrium point.
- The state may converge to one of several steady-state operatin points, depending on the initial state of the system. (Chapter 2, Example 2.2/2.3)
  First consider the dynamical systems

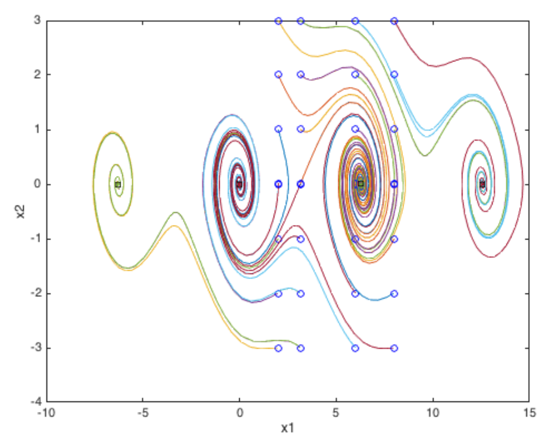
$$\dot{x}_1 = 0.5x_2 - 0.5h(x_1) \qquad \dot{x}_2 = -0.2x_1 - 0.3x_2 + 0.24$$

where $h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$, then we have 3 equilibrium points: $Q_1 = (0.0626, 0.7583)$, $Q_2 = (0.2854, 0.6097)$, $Q_3 = (0.8844, 0.2104)$
Second, consider the pendulum dynamics

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -\sin x_1 - 0.3x_2$$

then we have equilibrium points at $x_1 = n\pi, x_2 = 0$ for $n = 0, \pm 1, \pm 2, \cdots$



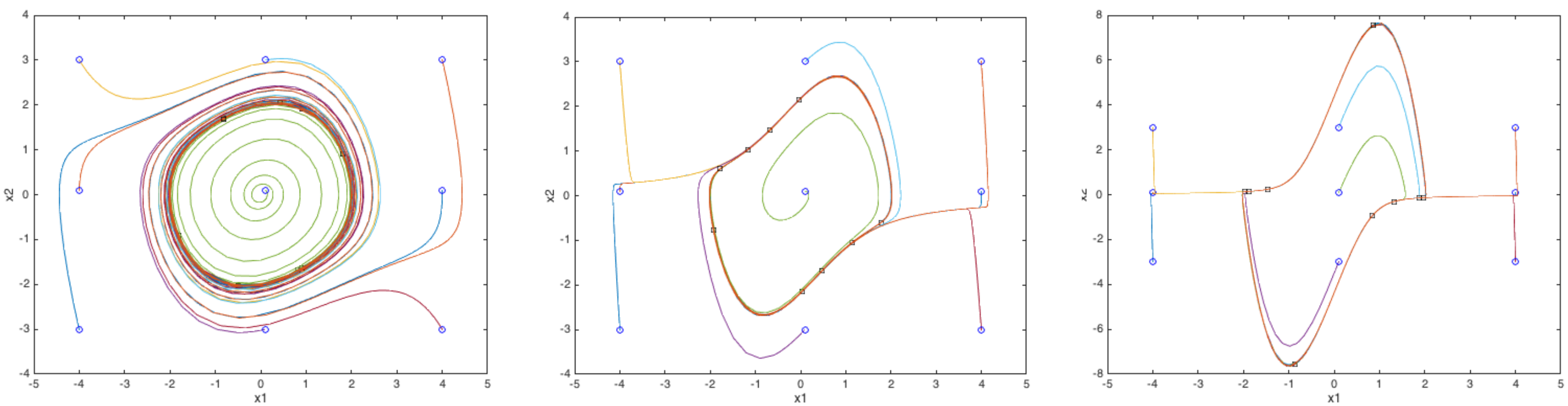


(Limit cycles)

- For a linear system to oscillate, it must have a pair of eigenvalues on the imaginary axis. The amplitude of oscillation will be dependent on the initial state.
- In real life, stable oscillation with fixed amplitude and frequency must be produced by nonlinear system, irrespective of the initial state. This type of oscillation is known as limit cycles. (Chapter 2, Example 2.4)
  Consider dynamics of Van der Pol oscillator

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2$$

  then it has only one isolated closed orbit.
  - When $\epsilon = 0.2$, the closed orbit is close to a circle of radius 2.
  - When $\epsilon = 1.0$, the circular shape of the closed orbit is distorted
  - When $\epsilon = 5.0$, the closed orbit is severely distorted.



(Subharmonic, harmonic, or almost-periodic oscillations)

- A stable linear system under a periodic input produces a periodic output of the same frequency. (basic principle of Bode plot)
- A nonlinear system under periodic excitation can oscillate with frequency that are submultiples or multiples of the input frequency

(Chaos)

- A nonlinear system can have a more complicated steady-state behavior that is not equilibrium or periodic oscillation. Such behavior is referred to as chaos
- Chaos may refer to any state of confusion or disorder.

(Multiple modes of behavior)

- An unforced system may have more than one limit cycle
- A forced system with periodic excitation may exhibit harmonic, subharmonic, or more complicated steady-state behavior, depending on the amplitude and frequency of the input.

- (HW # 1) solve 5 problems 1.1, 1.2, 1.4, 1.8, and 1.14, (If you want it, solve 1.12)