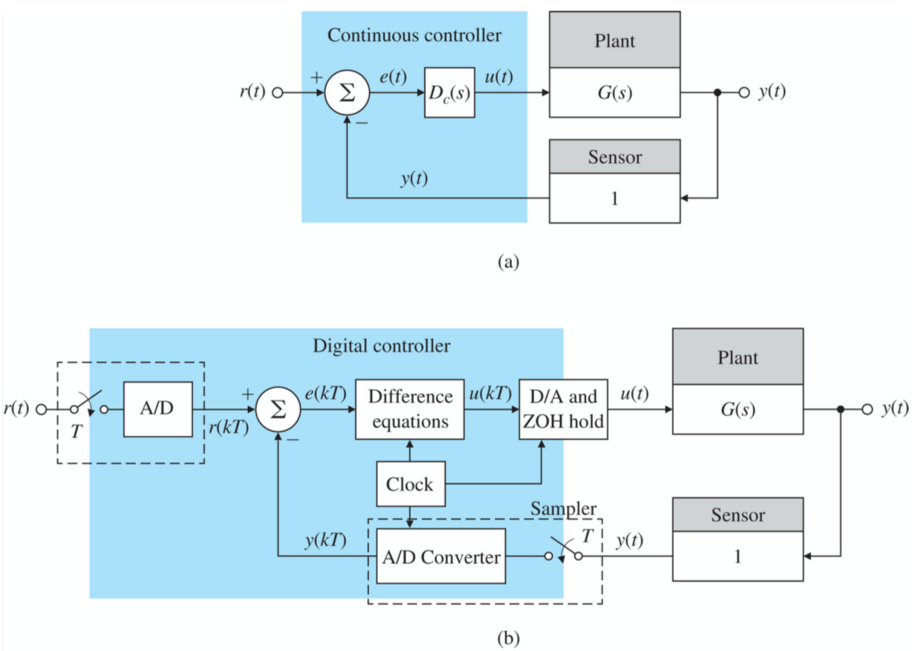
제 8 장

Digital Control

1 Digitization

1. Most control systems use digital computers (usually microprocessors) to implement the controller.
2. Sampler and A/D Converter, D/A Converter and ZOH (Zeroth-Order Holding), and Clock



3. The computation of error signal $e(t)$ and the dynamic compensation $D_c(s)$ can all be accomplished in a digital computer.

4. Difference equation for discrete time system $\leftrightarrow$ Differential equation for continuous time system

5. Two basic techniques for finding the difference equations for the digital controller, from $D_c(s)$ to $D_d(z)$

   - Discrete equivalent - section 8.3
   - Discrete design - section 8.7

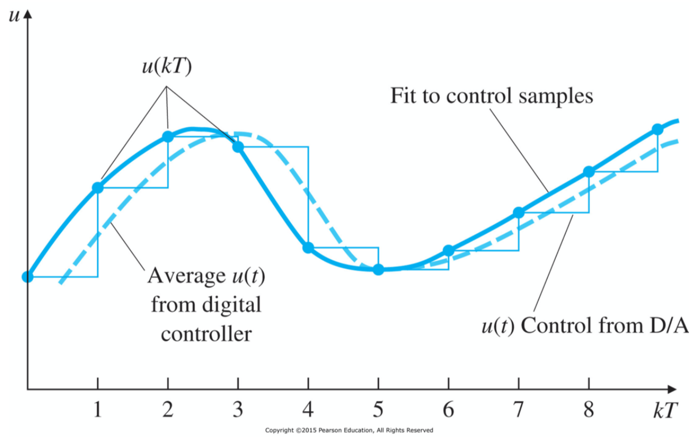
6. The analog output of the plant sensor is sampled and converted to a digital number in the analog-to-digital (A/D) converter. (Sampler and ADC)

   - Conversion from the continuous analog signal $y(t)$ to the discrete digital samples $y(kT)$ occurs repeatedly at instants of time T apart where T is the sample period $[s]$ and $1/T$ is the sample rate $[Hz]$.

     $$y(t) \qquad \rightarrow \qquad y(k) = y(kT) \quad \text{with} \quad t = kT$$

     where k is an integer and T is a fixed value.
   - The sampled signal is $y(kT)$, where k can take on any integer value.
   - It is often written simply as $y(k)$. We call this type of variable a discrete signal.

7. The D/A converter changes the digital binary number to an analog voltage, and a zeroth-order hold maintains the same voltage throughout the sample period T. (DAC and ZOH)



- Because each value of $u(kT)$ in Fig. 8.1(b) is held constant until the next value is available from the computer, the continuous value of $u(t)$ consists of steps (see Fig. 8.2) that, on average, are delayed from a fit to $u(kT)$ by $T/2$ as shown in the figure.
- Sample rates should be at least 20 times the bandwidth in order to assure that the digital controller will match the performance of the continuous controller.
- If we simply incorporate this $T/2$ delay into a continuous analysis of the system, an excellent prediction results in, especially, for sample rates much slower than 20 times bandwidth.

8. A system having both discrete and continuous signals is called a ‘sampled data system’.

2 Dynamic Analysis of Discrete Systems

- z-transform for discrete time systems $\leftrightarrow$ Laplace transform for continuous time systems.
- (8.2.1) z-Transform

1. Laplace transform and its important property

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}dt \qquad \mathcal{L}(\dot{f}(t)) = sF(s)$$

   where $f(0^+) = 0$

2. z-transform is defined by

$$\begin{aligned}\mathcal{Z}(f(k)) = F(z) &= \sum_{k=0}^{\infty} f(k)z^{-k}\\ &= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots\end{aligned}$$

$$\begin{aligned}\mathcal{Z}(f(k-1)) &= \sum_{k=0}^{\infty} f(k-1)z^{-k}\\ &= f(-1) + f(0)z^{-1} + f(1)z^{-2} + f(2)z^{-3} + \cdots\\ &= z^{-1}\left[f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots\right]\\ &= z^{-1}F(z)\end{aligned}$$

   where $f(k)$ is the sampled version of $f(t)$ and z^{-1} represents one sample delay, and $f(-1) = 0$.

3. Important property between LT and z-transform

$$z = e^{sT} \qquad \leftrightarrow \qquad s = \frac{1}{T}\ln z$$

4. For example, the general second-order difference equation

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k) + b_1 u(k-1) + b_2 u(k-2)$$

can be converted from this form to the z-transform of the variables $y(k)$ and $u(k)$ by invoking above relations,

$$Y(z) = (-a_1 z^{-1} - a_2 z^{-2}) Y(z) + (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z)$$

now we have a discrete transfer function:

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

- (8.2.2) z-Transform Inversion

1. See the Table 8.1 for understanding between z-transform and LT

$F(s)$	$f(kT)$	$F(z)$	
-	1, $k=0$ and 0, $k \neq 0$	1	
-	1, $k=k_0$ and 0, $k \neq k_0$	z^{-k_0}	
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$	$\frac{1}{1-z^{-1}}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$	$\frac{1}{1-e^{-aT}z^{-1}}$
$\frac{1}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$	$\frac{z^{-1}(1-e^{-aT})}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$	$\frac{z^{-1}\sin aT}{1-(2\cos aT)z^{-1}+z^{-2}}$
$\frac{s}{s^2+a^2}$	$\cos akT$	$\frac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$	$\frac{(1-z^{-1}\cos aT)}{1-(2\cos aT)z^{-1}+z^{-2}}$

2. For parts of Table, we have

$$\mathcal{Z}(\delta(t)) = 1 + 0z^{-1} + 0z^{-2} + \cdots = 1$$

$$\mathcal{Z}(\delta(t = k_0T)) = 0 + 0z^{-1} + \cdots + 1z^{-k_0} + \cdots + = z^{-k_0}$$

$$\mathcal{Z}(1(t)) = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1-z^{-1}} = (1-z^{-1})^{-1}$$

$$\mathcal{Z}(e^{-at}) = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + \cdots = \frac{1}{1-e^{-aT}z^{-1}}$$

3. The differentiator s is transformed into z-domain

$$\frac{1}{s} \quad \leftrightarrow \quad \frac{1}{1-z^{-1}} \qquad\qquad s \quad \leftrightarrow \quad (1-z^{-1})$$

4. z-transform of ramp signal $t = kT$ becomes

$$\begin{aligned}
\mathcal{Z}(t) &= 0 + Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \cdots \\
&= T[z^{-1} + 2z^{-2} + 3z^{-3} + \cdots] \\
z^{-1}\mathcal{Z}(t) &= T[z^{-2} + 2z^{-3} + 3z^{-4} + \cdots] \\
(1-z^{-1})\mathcal{Z}(t) &= T[z^{-1} + z^{-2} + z^{-3} + \cdots] = T\frac{z^{-1}}{1-z^{-1}} \\
\mathcal{Z}(t) &= \frac{Tz^{-1}}{(1-z^{-1})^2}
\end{aligned}$$

5. A z-transform inversion technique that has no continuous counterpart is called 'long division'. For example, consider a first-order discrete system

$$y(k) = \alpha y(k-1) + u(k) \qquad \rightarrow \qquad \frac{Y(z)}{U(z)} = \frac{1}{1-\alpha z^{-1}}$$

For a unit-pulse input, its z-transform is

$$U(z) = 1$$

so the long division becomes

$$\begin{aligned} Y(z) &= \frac{1}{1-\alpha z^{-1}} \\ &= 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3} \cdots \end{aligned}$$

We see that the sampled time history of y is

$$y(0) = 1 \qquad y(1) = \alpha \qquad y(2) = \alpha^2 \qquad y(3) = \alpha^3 \qquad \cdots$$

- (8.2.3) Relationship between s and z

1. Consider the continuous signal of

$$f(t) = e^{-at} \qquad t > 0$$
$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = \frac{1}{s+a}$$

   and it corresponds to a pole $s = -a$.

2. Consider the discrete signal of

$$f(kT) = e^{-akT}$$
$$F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k} = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + e^{-3aT}z^{-3} + \cdots \qquad \text{무한등비급수}$$
$$= \frac{\text{초기치}}{1 - \text{공비}} = \frac{1}{1 - e^{-aT}z^{-1}} = \frac{z}{z - e^{-aT}}$$

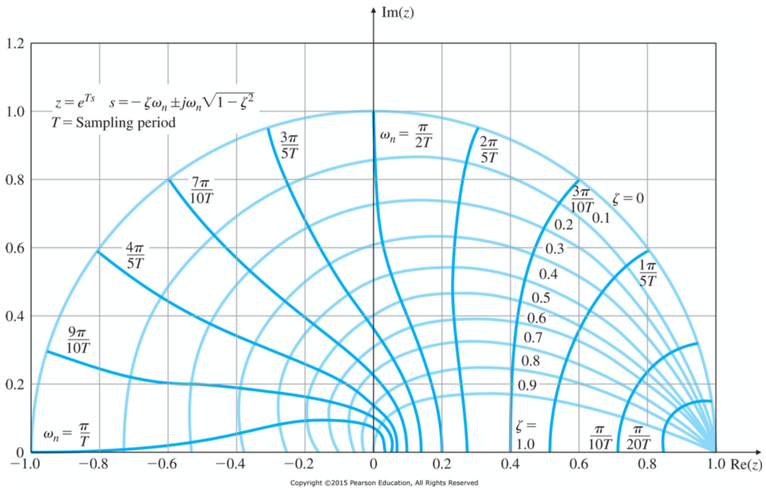
   and it corresponds to a pole $z = e^{-aT}$.

3. The equivalent characteristics in the z-plane are related to those in the s-plane by the expression

$$z = e^{sT} = e^{-aT+jbT} = e^{-aT}(\cos bT + j\sin b)$$
$$= e^{-\sigma T}(\cos\omega_d T + j\sin\omega_d T)$$
$$= e^{-\zeta\omega_n T}(\cos\omega_n\sqrt{1-\zeta^2}T + j\sin\omega_n\sqrt{1-\zeta^2}T)$$

   where T is the sample period, and $s = -\sigma + j\omega_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$

4. See Fig. 8.4, and it shows the mapping of lines of constant damping ζ and natural frequency ω_n from s-plane to the upper half of the z-plane, using $z = e^{sT}$.



a) The stability boundary $s = 0 \pm j\omega$ becomes the unit circle $|z| = 1$ in the z-plane; inside the unit circle is stable, outside is unstable

b) The small vicinity around $z = +1$ in the z-plane is essentially identical to the vicinity around the origin $s = 0$, in the s-plane.

c) The z-plane locations give response information normalized to the sample rate rather than to time as in the s-plane.

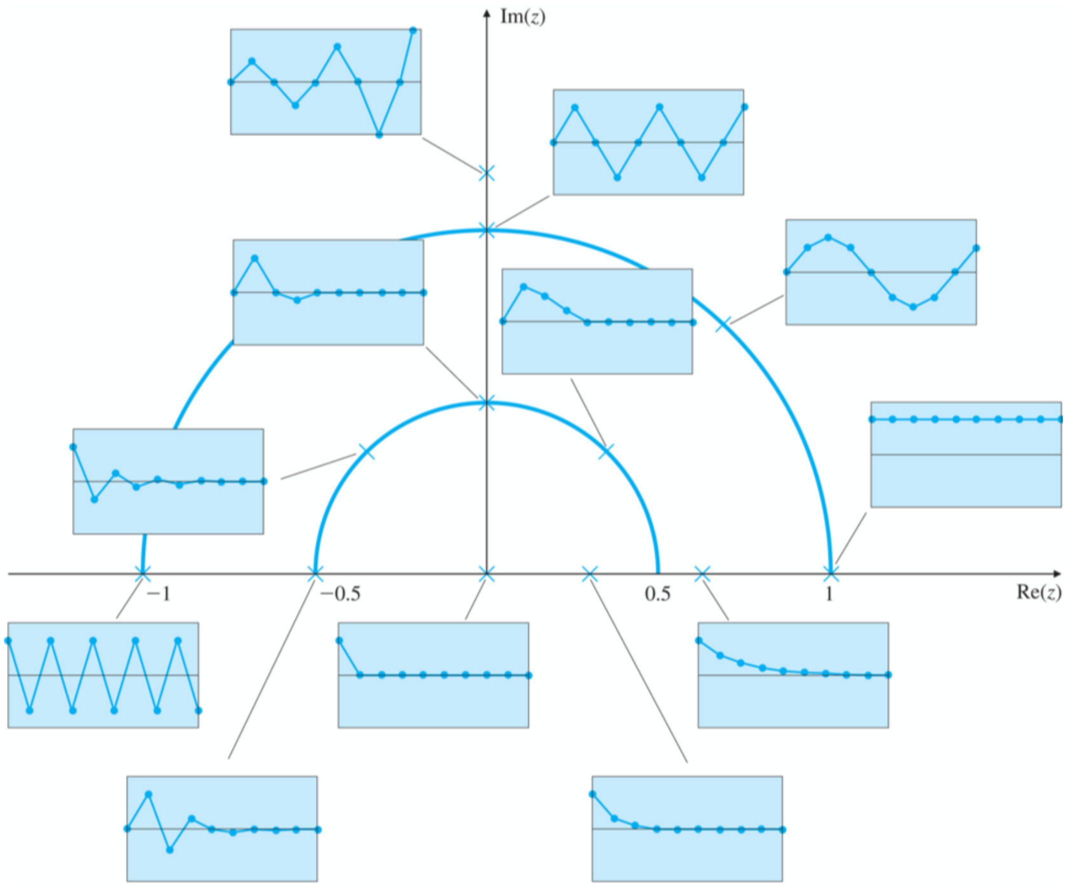
d) The negative real z-axis always represents a frequency of $\omega_s/2$, where $\omega_s = 2\pi/T$ = circular sample rate in radians per second.

e) Vertical lines in the left half of the s-plane (the constant real part of s) map into *circles* within the unit circle of the z-plane

f) Horizontal lines in the s-plane (the constant imaginary part of s) map into *radial lines* in the z-plane.

g) Frequencies greater than $\omega_s/2$, called the Nyquist frequency, appear in the z-plane on the top of corresponding lower frequencies because of the circular characteristics of e^{sT}. This overlap is called *aliasing* or folding.

5. As a result, it is necessary to sample at least twice as fast as a signal's highest frequency component in order to represent that signal with the samples.
6. The figure sketches time responses that would result from poles at the indicated locations.



- (8.2.4) Final Value Theorem

  1. Discrete final value theorem is

$$\lim_{t\to\infty} x(t) = x_{ss} = \lim_{s\to 0} sX(s) \qquad\qquad \lim_{k\to\infty} x(k) = x_{ss} = \lim_{z\to 1}(1 - z^{-1})X(z)$$

     if all the poles of $(1 - z^{-1})X(z)$ are inside the unit circle.

  2. For example, to find the DC gain of the TF

$$G(z) = \frac{X(z)}{U(z)} = \frac{0.58(1+z)}{z+0.16}$$

     we let $u(k) = 1$ for $k \geq 0$, so that

$$U(z) = \frac{1}{1 - z^{-1}}$$

     and

$$X(z) = \frac{0.58(1+z)}{(1 - z^{-1})(z + 0.16)}$$

     Applying the final value theorem yields

$$x_{ss} = \lim_{z\to 1}(1 - z^{-1})X(z) = \frac{0.58 \cdot 2}{1 + 0.16} = 1$$

     so the DC gain of $G(z)$ is unity.