(Revisited Example 3.29) Determine the effect of the complex zero locations ($s = -\alpha \pm j\beta$) on the unit-step response of the system for three different zero locations using MATLAB
$(\alpha, \beta) = (0.1, 1.0), (0.25, 1.0), (0.5, 1.0)$

$$H(s) = \frac{(s+\alpha)^2 + \beta^2}{(s+1)[(s+0.1)^2+1]}$$

Figure 3.33 Locations of complex zeros

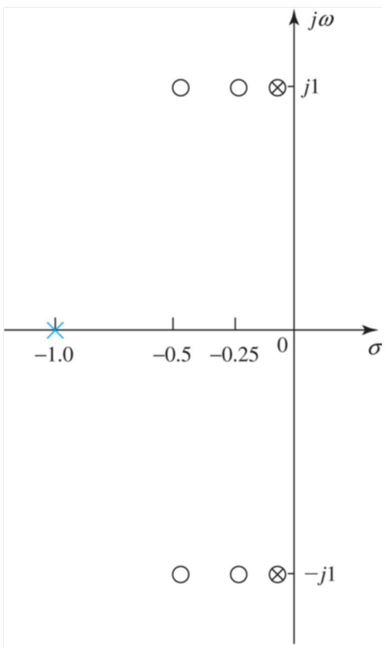
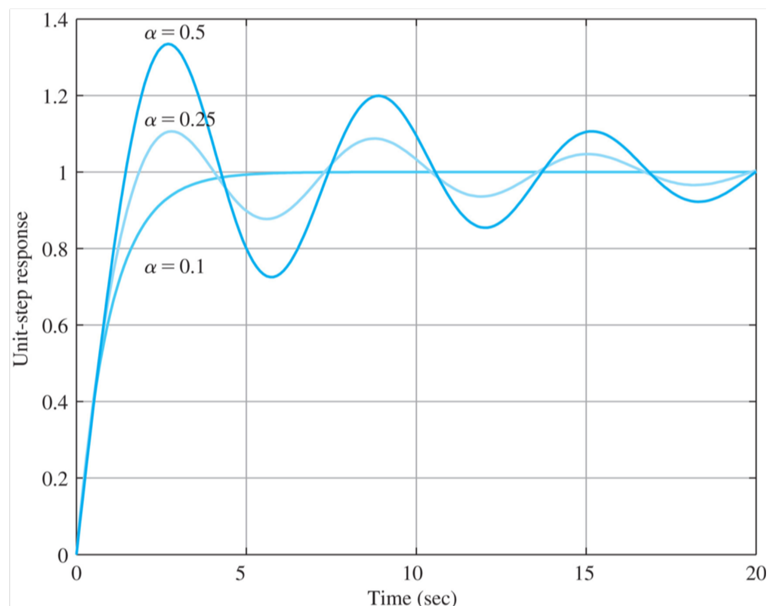


Figure 3.34 Effect of complex zeros on transient response



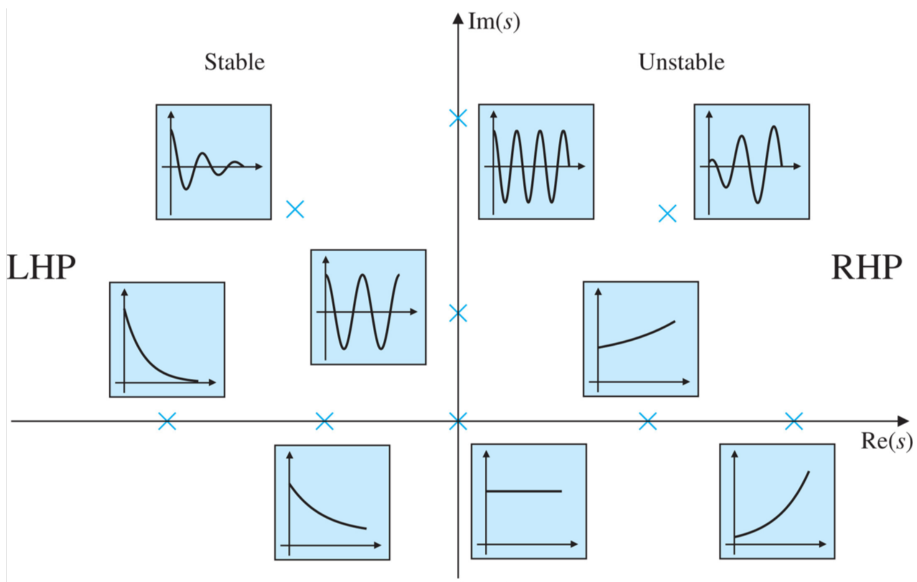
6 Stability

1. An LTI system is said to be "stable" if all the roots of the TF denominator polynomial have negative real parts (that is, they are all in the LHP) and is "unstable" otherwise.

   Consider the location of pole

   $$s = -\sigma \pm j\omega$$

   - Stable, if all the poles have negative real parts $\sigma > 0$
   - Unstable, if any pole of the system is in the RHP, $\sigma < 0$
   - Check whether the poles are repeated on the $j\omega$ axis, if $\sigma = 0$.



2. Bounded Input-Bounded Output (BIBO) Stability

- A system is said to have bounded input-bounded output (BIBO) stability if every bounded input results in a bounded outout (regardless of what goes on inside the system).
- Consider the output

$$y(t) = \int_0^\infty h(\tau)u(t-\tau)d\tau$$

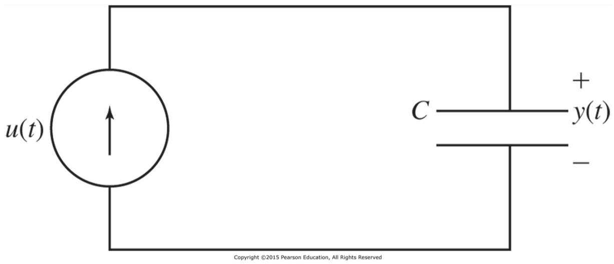
If $|u(t)| < M < \infty$ is bounded, then the output is bounded by

$$\begin{aligned} |y| &= \left| \int_0^\infty hud\tau \right| \\ &\leq \int_0^\infty |h||u|d\tau \\ &\leq M \int_0^\infty |h(\tau)|d\tau \end{aligned}$$

Thus the output will be bounded if $\int_0^\infty |h(\tau)|d\tau$ is bounded.

- The system with impulse response $h(t)$ is "BIBO-stable" if and only if the integral

$$\int_0^\infty |h(\tau)|d\tau < \infty$$



- (Example 3.31, BIBO Stability for a Capacitor) Consider the capacitor voltage as output $y(t)$ and current as input $u(t)$

$$C\frac{dy(t)}{dt} = u(t) \qquad \text{with} \quad C = 1$$

The TF and its impulse response are

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s} \qquad\qquad h(t) = 1(t)$$

Since $\int_0^\infty 1 d\tau = \infty$ is not bounded, the capacitor is not BIBO-stable.

  - If an LTI system has any pole on the imaginary axis or in the RHP, the response will not be BIBO-stable.
  - If every pole is inside the LHP, then the response will be BIBO-stable.

3. Stability of LTI Systems

- Stable or Unstable according to the locations of poles, whether all the poles are inside LHP or any pole is in RHP.
- Neutrally stable if the system has "non-repeated $j\omega$ axis poles".
  - A pole at the origin (an integrator) results in a non-decaying transient
  - A pair of complex $j\omega$ axis poles results in an oscillating response with constant amplitude.
- Unstable if the system has "repeated poles on the $j\omega$ axis" because it brings in $te^{\pm j\omega_i t}$ terms.

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1(t) \qquad \mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t \qquad \mathcal{L}^{-1}\left[\frac{1}{(s+a)^2}\right] = te^{-at}$$

4. Routh' Stability Criterion

- There are several methods of obtaining information about the locations of the roots of a polynomial w/o actually solving the roots.
- Consider the characteristic equation of nth-order system of $G(s) = \frac{b(s)}{a(s)}$:

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

- A necessary condition for stability is that all the coefficients of the characteristic polynomial be positive

$$a_1 > 0 \quad a_2 > 0 \quad \cdots \quad a_{n-1} > 0 \quad a_n > 0$$

- A system is "stable" if and only if all the elements in the first column of the Routh array are positive

$$\begin{array}{rlllll}
s^n : & 1 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} : & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} : & b_1 & b_2 & b_3 & \cdots & \\
s^{n-3} : & c_1 & c_2 & c_3 & \cdots & \\
\vdots : & \vdots & \vdots & & & \\
s^1 : & * & * & & & \\
s^0 : & * & & & &
\end{array}$$

where

$$b_1 = \frac{a_1 a_2 - a_3}{a_1} \qquad b_2 = \frac{a_1 a_4 - a_5}{a_1} \qquad b_4 = \frac{a_1 a_6 - a_7}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

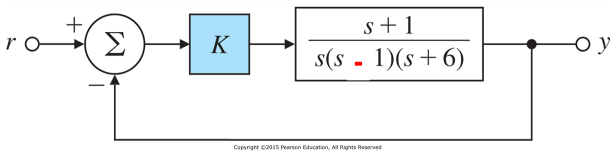
- If the elements of the first column are not all positive, then the number of roots in the RHP equals the number of sign changes in the column.

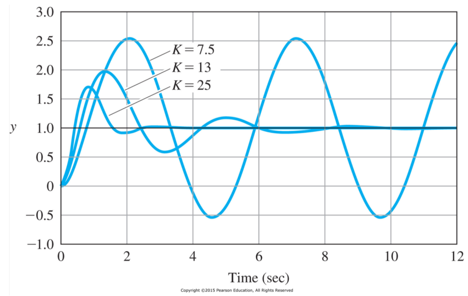
- (Example 3.32 Routh's Test) Determine the number of poles in the RHP w/o solving the roots

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$$\begin{array}{llll}
s^6 : 1 & 3 & 1 & 4 \\
s^5 : 4 & 2 & 4 & \\
s^4 : \dfrac{12-2}{4} = 2.5 & \dfrac{4-4}{4} = 0 & 4 & \\
s^3 : \dfrac{5-0}{2.5} = 2 & \dfrac{10-16}{2.5} = -2.4 & & \\
s^2 : \dfrac{0+6}{2} = 3 & 4 & & \\
s^1 : \dfrac{-7.2-8}{3} = -8.067 & & & \\
s^0 : 4 & & &
\end{array}$$

There are "two poles in the RHP" b/c there are two sign changes in the first column.





- (Example 3.33, Stability vs Parameter Range) Determine the range of K over which the system is stable ?
  The characteristic equation is:

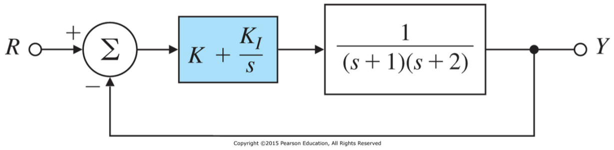
$$1 + K\frac{s+1}{s(s-1)(s+6)} = 0 \qquad \rightarrow \qquad s^3 + 5s^2 + (K-6)s + K = 0$$

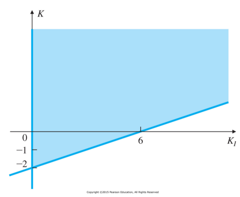
where $K > 6$ is required as necessary condition. The corresponding Routh array is

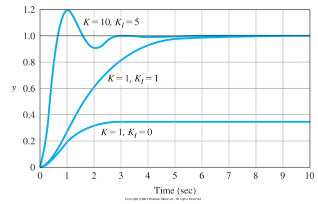
$$\begin{array}{ll} s^3 : 1 & K-6 \\ s^2 : 5 & K \\ s^1 : \dfrac{5(K-6)-K}{5} & \\ s^0 : K & \end{array}$$

For the system to be stable, it is necessary that

$$\frac{5(K-6)-K}{5} > 0 \quad \text{and} \quad K > 0 \qquad \rightarrow \qquad \therefore \quad K > 7.5$$







- (Example 3-34, Stability vs Two Parameter Ranges) Find the range of the controller gains so that the PI feedback system is stable ?
  The characteristic equation is

$$1 + \left(K + \frac{K_I}{s}\right)\frac{1}{(s+1)(s+2)} = 0 \qquad \rightarrow \qquad s^3 + 3s^2 + (2+K)s + K_I = 0$$

  The corresponding Routh array is

$$\begin{array}{ll} s^3 : 1 & K+2 \\ s^2 : 3 & K_I \\ s^1 : \dfrac{3(K+2) - K_I}{3} & \\ s^0 : K_I & \end{array}$$

  For the stability, we must have the allowable region as follow:

$$K_I > 0 \quad \text{and} \quad K > \frac{1}{3}K_I - 2$$

  - For $K = 1$ and $K_I = 0$, closed-loop poles are at 0 and $-1.5 \pm 0.86j$
  - For $K = 1$ and $K_I = 1$, closed-loop poles are all -1
  - For $K = 10$ and $K_I = 5$, closed-loop poles are at -0.46 and $-1.26 \pm 3.3j$

1. (Homework #3, Due : May 16, 2020) Solve and Submit 3.2, 3.4, 3.7, 3.9, 3.12, 3.17, 3.20, 3.24, 3.26, 3.32, 3.33, 3.35, 3.36, 3.37, 3.40, 3.51, 3.53, 3.56, 3.60