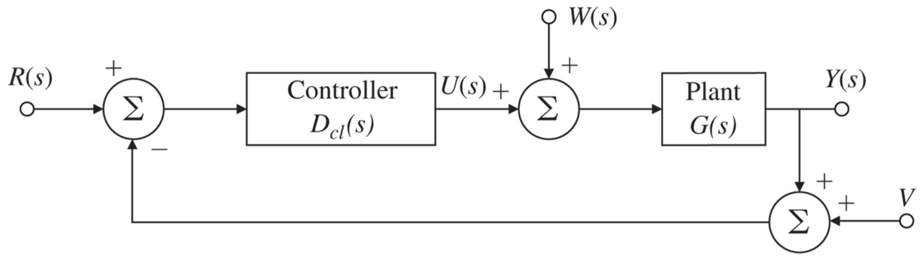
2 Steady-State Error to Polynomial Inputs: System Type

- Reference for Tracking : $r(t) = a_0 t^n + a_1 t^{n-1} + a_2 t^{n-2} + \cdots + a_{n-1} t + a_n$
- Reference for Regulation : $r = constant$
- The system type is defined as the "degree of the polynomial that it can reasonably track".
  e.g., a system that can track a polynomial of degree 1 with a constant error is called Type 1.

1. System Type for Tracking
   - Reference polynomial inputs:
     - step (position) input : $r(t) = 1(t)$ for $t \geq 0$ and $R(s) = \frac{1}{s}$
     - ramp (velocity) input : $r(t) = t \cdot 1(t)$ for $t \geq 0$ and $R(s) = \frac{1}{s^2}$
     - parabolic (acceleration) input : $r(t) = \frac{1}{2}t^2 \cdot 1(t)$ for $t \geq 0$ and $R(s) = \frac{1}{s^3}$
     - k-th order: $r(t) = \frac{t^k}{k!}$ for $t \geq 0$ and $R(s) = \frac{1}{s^{k+1}}$



- **Error TF** $E(s) = R(s) - Y(s)$ **with** $V = W = 0$

$$E(s) = \frac{1}{1 + GD_{cl}(s)} R(s) = S(s)R(s) \quad \text{with the definition of sensitivity function} \quad S(s) = \frac{1}{1 + GD_{cl}(s)}$$

- **Steady-state error for k-th order polynomial inputs** $r(t) = \frac{t^k}{k!}$ **is**

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} s \frac{1}{1 + GD_{cl}(s)} \frac{1}{s^{k+1}} \end{aligned}$$

- **Let us collect the all the terms except the pole(s) at the origin into a function** $GD_{clo}(s)$**, which is finite at** $s = 0$ **so that we can define the constant** $GD_{clo}(0) = K_n$ **and write the loop TF as**

$$GD_{cl}(s) = \frac{GD_{clo}(s)}{s^n}$$

For example, if $GD_{cl}(s)$ **has no integrator, then** $n = 0$**. If the system has one integrator, then**

$n = 1$, and so forth. Substituting this expression into above equation:

$$
\begin{aligned}
e_{ss} &= \lim_{s \to 0} s \frac{1}{1 + \frac{GD_{clo}(s)}{s^n}} \frac{1}{s^{k+1}} \\
&= \lim_{s \to 0} \frac{s^n}{s^n + K_n} \frac{1}{s^k}
\end{aligned}
$$

- Step input $k = 0$,
  - if $n = 0$, then $e_{ss} = \frac{1}{1+K_0}$ and the system type 0
    where $K_0 = \lim_{s \to 0} GD_{clo}(s) \quad \to \quad K_p = \lim_{s \to 0} GD_{cl}(s)$ is called as "position error constant"
  - if $n = 1$, then $e_{ss} = 0$
  - if $n = 2$, then $e_{ss} = 0$
- Ramp input $k = 1$,
  - if $n = 0$, then $e_{ss} = \infty$
  - if $n = 1$, then $e_{ss} = \frac{1}{K_1}$ and the system type 1
    where $K_1 = \lim_{s \to 0} GD_{clo}(s) \quad \to \quad K_v = \lim_{s \to 0} sGD_{cl}(s)$ is "velocity error constant"
  - if $n = 2$, then $e_{ss} = 0$
- Parabolic input $k = 2$,
  - if $n = 0$, then $e_{ss} = \infty$
  - if $n = 1$, then $e_{ss} = \infty$
  - if $n = 2$, then $e_{ss} = \frac{1}{K_2}$ and the system type 2
    where $K_2 = \lim_{s \to 0} GD_{clo}(s) \quad \to \quad K_a = \lim_{s \to 0} s^2 GD_{cl}(s)$ is "acceleration error constant"

- The type information can also be usefully gathered in a table of error values as a function of the degree of the input polynomial and the type of the system as shown in Table 4.1

Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

- (Example 4.1 System Type for Speed Control) Assume $G(s) = \frac{A}{\tau s+1}$ and $D_{cl} = k_p$. In this case,

$$GD_{cl} = \frac{Ak_p}{\tau s + 1}$$

is "type 0" and its position constant is $K_p = Ak_p$ and the steady-state error is $e_{ss} = \frac{1}{1+Ak_p}$ for step input. Also we can check it again from the definition of steady-state error according to the inputs (step, ramp, parabolic):

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+GD_{cl}}\frac{1}{s} = \frac{1}{1+Ak_p} \qquad \text{type } 0$$
$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+GD_{cl}}\frac{1}{s^2} = \infty$$
$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+GD_{cl}}\frac{1}{s^3} = \infty$$

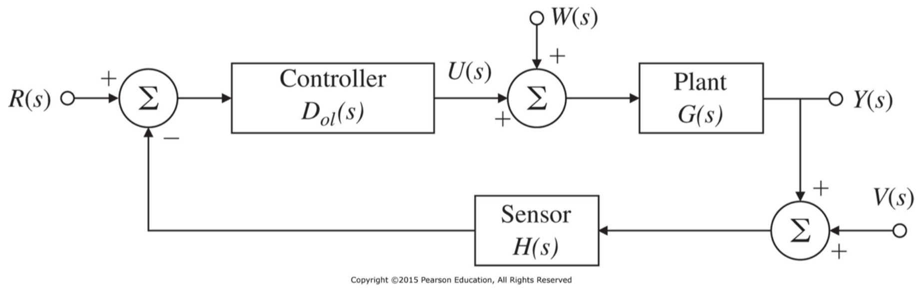
- (Example 4.2 System Type using PI Control) Assume $G(s) = \frac{A}{\tau s+1}$ and $D_{cl} = k_P + \frac{k_I}{s} = \frac{k_P s + k_I}{s}$. In this case,

$$GD_{cl} = \frac{A(k_P s + k_I)}{s(\tau s + 1)}$$

is "type 1" and its velocity constant is $K_v = Ak_I$ and the steady-state error is $e_{ss} = \frac{1}{Ak_I}$ for ramp input. Also we can check it again from the definition of steady-state error according to the inputs (step, ramp, parabolic):

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s \frac{1}{1+GD_{cl}} \frac{1}{s} = 0 \\ e_{ss} &= \lim_{s\to 0} s \frac{1}{1+GD_{cl}} \frac{1}{s^2} = \frac{1}{Ak_I} \qquad \text{type 1} \\ e_{ss} &= \lim_{s\to 0} s \frac{1}{1+GD_{cl}} \frac{1}{s^3} = \infty \end{aligned}$$

- The system type is a "robust property with respect to parameter changes in the unity feedback" structure
- As the system type increases, so the tracking performance is improved.



- (Example 4.3 System Type for a Servo with Tachometer Feedback) Consider Fig. 4.5 with $G(s) = \frac{1}{s(\tau s+1)}$, $D_c(s) = k_P$ and $H(s) = 1 + k_t s$. Determine the system type and relevant error constant with respect to the reference inputs when $V = W = 0$.
  The error TF is

$$\begin{aligned} E(s) &= R(s) - \frac{D_c G}{1 + D_c G H} R(s) \\ &= \left[1 - \frac{\frac{k_P}{s(\tau s+1)}}{1 + \frac{k_P(1+k_t s)}{s(\tau s+1)}}\right] R(s) = \left[1 - \frac{k_P}{s(\tau s + 1) + k_P(1 + k_t s)}\right] R(s) \\ &= \left[1 - \frac{k_P}{\tau s^2 + (k_P k_t + 1)s + k_p}\right] R(s) = \frac{\tau s^2 + (k_P k_t + 1)s}{\tau s^2 + (k_P k_t + 1)s + k_p} R(s) \end{aligned}$$

  The steady-state error for inputs (step, ramp, parabolic) becomes

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s \frac{s[\tau s + (k_P k_t + 1)]}{\tau s^2 + (k_P k_t + 1)s + k_p} \frac{1}{s} = 0 \\ e_{ss} &= \lim_{s\to 0} s \frac{s[\tau s + (k_P k_t + 1)]}{\tau s^2 + (k_P k_t + 1)s + k_p} \frac{1}{s^2} = \frac{k_P k_t + 1}{k_p} \qquad \text{type 1} \\ e_{ss} &= \lim_{s\to 0} s \frac{s[\tau s + (k_P k_t + 1)]}{\tau s^2 + (k_P k_t + 1)s + k_p} \frac{1}{s^3} = \infty \end{aligned}$$

Thus, the “system type is 1” and its velocity constant becomes

$$K_v = \frac{k_p}{k_P k_t + 1}$$

- If tachometer feedback is used to improve dynamic response, the steady-state error is usually increased, that is, there is a trade-off between improving the stability and reducing steady-state error.

2. System Type for Regulation and Disturbance Rejection

- A system can also be classified with respect to its ability to reject polynomial disturbance inputs in a way analogous to the classification scheme based on reference inputs.
- Consider TF from the disturbance input $W(s)$ to the error $E(s)$ when $R(s) = 0$

$$\frac{E(s)}{W(s)} = \frac{R(s) - Y(s)}{W(s)} = -\frac{Y(s)}{W(s)} = T_w(s)$$

  - "Type 0" if step disturbance input results in a nonzero constant steady-state error
  - "Type 1" if ramp disturbance input results in a nonzero constant steady-state error
  - "Type 2" if parabolic disturbance input results in a nonzero constant steady-state error

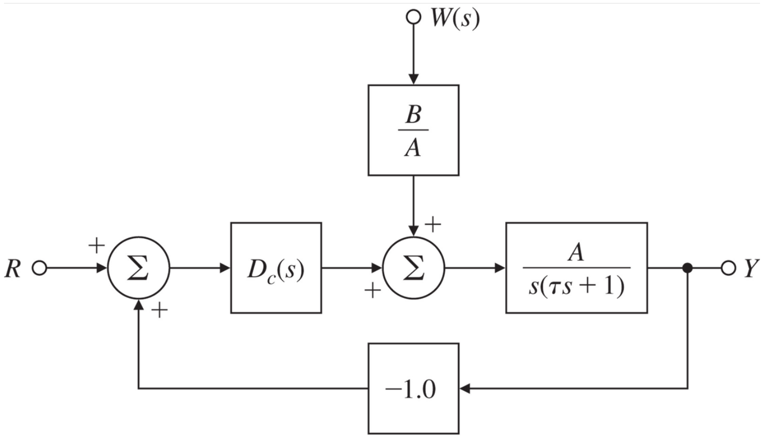
- Assume that a constant n and a function $T_{o,w}(s)$ can be defined with the properties that $T_{o,w}(0) = \frac{1}{K_{n,w}}$ and that the disturbance-to-error TF can be written as

$$T_w(s) = s^n T_{o,w}(s)$$

  Then the steady-state error to disturbance input, which a polynomial of degree k, is

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s T_w(s) \frac{1}{s^{k+1}} \\ &= \lim_{s \to 0} T_{o,w}(s) \frac{s^n}{s^k} \end{aligned}$$

  - if $n > k$, then the error is zero
  - if $n = k$, the system is "type k" and the error is given by $\frac{1}{K_{n,w}}$
  - if $n < k$, the error is unbounded.



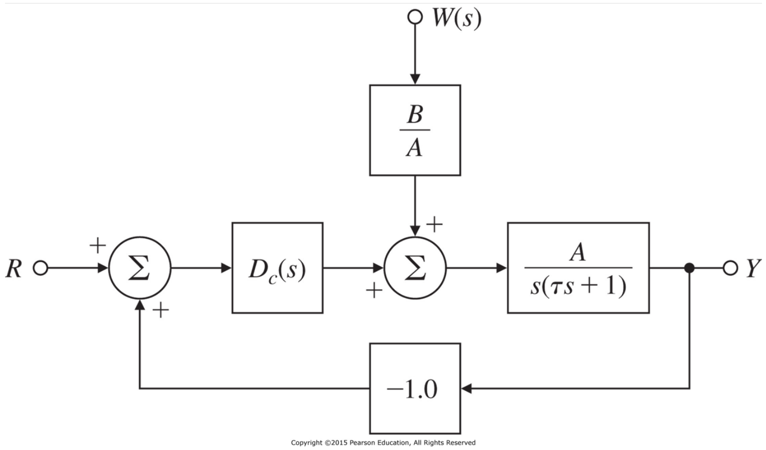
- (Example 4.4 System Type for a DC Motor Position Control) Consider Fig. 4.6 with $R(s) = 0$, (a) $D_c(s) = k_P$, (b) $D_c(s) = k_P + \frac{k_I}{s} = \frac{k_P s + k_I}{s}$

  a) Proportional control $D_c(s) = k_P$

$$\begin{aligned} T_w(s) &= -\frac{\frac{B}{s(\tau s+1)}}{1 + \frac{Ak_P}{s(\tau s+1)}} \\ &= -\frac{B}{\tau s^2 + s + Ak_P} \\ &= -s^0 T_{o,w}(s) \qquad \rightarrow \qquad \text{type } 0 \end{aligned}$$

$$e_{ss} = -\lim_{s\to 0} s \frac{B}{\tau s^2 + s + Ak_P} \frac{1}{s} = -\frac{B}{Ak_P} \qquad \rightarrow \qquad K_{0,w} = \frac{1}{T_{o,w}(0)} = -\frac{Ak_P}{B}$$

  where unit-step disturbance brings a non-zero constant error, unit-ramp and unit-parabolic disturbances yield unbounded error.



b) Proportional-Integral control $D_c(s) = \frac{k_P s + k_I}{s}$

$$
\begin{aligned}
T_w(s) &= -\frac{\frac{B}{s(\tau s+1)}}{1+\frac{A(k_P s+k_I)}{s^2(\tau s+1)}} \\
&= -\frac{Bs}{\tau s^3 + s^2 + Ak_P s + Ak_I} \\
&= -s^1 T_{o,w}(s) \qquad \rightarrow \qquad \text{type 1} \\
e_{ss} &= -\lim_{s\to 0} s\frac{Bs}{\tau s^3 + s^2 + Ak_P s + Ak_I}\frac{1}{s^2} = -\frac{B}{Ak_I} \qquad \rightarrow \qquad K_{1,w} = \frac{1}{T_{o,w}(0)} = -\frac{Ak_I}{B}
\end{aligned}
$$

where unit-step disturbance brings a zero error, unit-ramp disturbance yields a non-zero constant error, and unit-parabolic disturbance produces unbounded error.

- As the system type increases, so the robustness against disturbance is improved.