STANDARD $H_\infty$ STATE-SPACE SOLUTION BY YOULA PARAMETERIZATION

Youngjin Choi*, Wan Kyun Chung*, Il Hong Su† and Sang Rok Oh‡

* Robotics Lab. School of Mechanical Engineering
Pohang University of Science and Technology (POSTECH), KOREA.
yjchoi@risbot.rist.re.kr or wkchung@vision.postech.ac.kr
† INCOR Lab. Department of Electrical Engineering
HanYang University, KOREA.
‡ Division of Electrical and Information Technology, KIST, KOREA.

Abstract: In this paper, we propose one methodology to parameterize all stabilizing controllers from the central controller. This is performed using the so called "constrained doubly coprime factorization" for the constrained plant obtained from $H_\infty$ norm constraint and central controller. The constrained left and right coprime factors are obtained by Bezout Identity, and these are used to provide Youla parameterization in which $H_\infty$ norm of closed-loop transfer matrix can be smaller than any prescribed value. This result shows the connection between two different method of standard state-space $H_\infty$ solution by inner function and Youla parameterization by constrained doubly coprime factorization.

Keywords: Youla parameterization, central controller, doubly coprime factorization.

1 INTRODUCTION

The parameterization of stabilizing controllers was first introduced by Youla et al(1976). Especially, the Youla parameterization provides a systematic way to choose the stabilizing controllers. In conventional Youla parameterization approach to $H_\infty$ control, matrix dilation optimization and Hankel norm approximation plays an important role in that $H_\infty$ norm of closed-loop transfer matrix can be constrained smaller than design parameter $\gamma$, and the controller designed by this method has 3-times larger dimension than the system state dimension. To overcome these difficulties, Doyle et al(1989) suggested standard state-space $H_\infty$ solution by using the concept of inner function and internal stability.

Section 2 defines the problem, section 3 introduces conventional Youla parameterization as a preliminary. In section 4, we show that the state error dynamics is asymptotically stable by using central controller. The concept of constrained plant is introduced and the doubly coprime factorization for it is derived from Bezout Identity. All stabilizing controllers are expressed with free parameter. Section 5 concludes the paper.

For future notations, the Hardy space of stable and proper function is expressed by $RH_\infty$, which denotes analytic function in right half region of complex plane. The spectral radius is defined as

$$\rho(A) = \max_i |\lambda_i(A)|, \quad (1)$$

where $A$ is $n \times n$ matrix and $\lambda$ denotes an eigenvalue. The lower linear fractional transformation of $P$ on $K$ is represented by

$$F_L(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}, \quad (2)$$

where $I$ is an identity matrix of suitable dimension.

2 PROBLEM STATEMENT

Doubly coprime factorization has been used to parameterize all stabilizing controllers from any cen-
tral controller. However, the standard state-space $\mathcal{H}_\infty$ controller (Doyle et al., 1989) cannot be derived from the conventional doubly coprime factorization because it uses the concept of inner function to parameterize all stabilizing controllers. On the contrary, the central controller can not be derived from Youla parameterization approach to $\mathcal{H}_\infty$ control. The primary goal of this paper is to develop another doubly coprime factorization to parameterize all stabilizing controllers from the central controller which satisfies $\mathcal{H}_\infty$ norm constraint. By using it, we will derive the standard state-space $\mathcal{H}_\infty$ solution with free parameter.

Although, in this paper, we start with the central controller of Doyle et al. (1989), this central controller can be obtained by other method such as the minimum entropy solution (Zhou, 1996; Glover, 1988). This means that if we have only central controller by any methods, we can generate standard state-space $\mathcal{H}_\infty$ solution with free parameter using the proposed method in this paper. This will provide the connection between the two different approaches.

3 PRELIMINARY:
Conventional (Unconstrained) Youla Parameterization

For a given plant $G(s) = C_2(sI - A)^{-1}B_2 + D_{22}$, the generalized plant can be expressed by

$$ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} $$

(3)

Assume that the following properties are satisfied for a given system (3) to describe controller as simple as possible:

1. $(A, B_2)$ is stabilizable.
2. $(A, C_2)$ is detectable.
3. $D_{12}^T [C_1 \ D_{12}] = [0 \ I]$.
4. $[B_1 \ D_{21}]^T D_{21} = [0 \ I]$.
5. $D_{11} = 0$ and $D_{22} = 0$.

where 0 is zero matrix of suitable dimension. If the above five assumptions are not satisfied for the generalized plant, the loop transformation and loop scaling (Green, 1995) may be needed to satisfy these assumptions. Since $\mathcal{H}_\infty$ central controller will be used to parameterize all stabilizing controllers, we utilize following equations for generalized plant satisfying the above assumptions without derivation and proof because these can be found in many references (Doyle, 1989; Zhou, 1995):

$$ X_\infty = \text{Ric} \left[ \begin{array}{c}
A \\
-\gamma^2 B_1 B_1^T - B_2 B_2^T \\
-\gamma^{-2} C_1^T C_1 - C_2^T C_2 \\
\end{array} \right] $$

(4)

$$ Y_\infty = \text{Ric} \left[ \begin{array}{c}
A^T \\
-\gamma^{-2} B_1 B_1^T \\
-\gamma^{-2} C_1^T C_1 - C_2^T C_2 \\
\end{array} \right] $$

(5)

$$ F_\infty = -B_2^T X_\infty $$

$$ H_\infty = -Y_\infty C_2^T $$

$$ Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1} $$

where $X_\infty \geq 0, Y_\infty \geq 0$ and $\rho(X_\infty Y_\infty) < \gamma^2$. The solution of Riccati equation is real, symmetric and semi-positive definite matrix. When $\gamma \to \infty$, we define $X_\infty = X_\infty, Y_\infty = Y_\infty$ and matrix $Z_\infty$ is equal to identity. Also, we define $F_\infty = -B_2^T X_\infty$ and $H_\infty = -Y_\infty C_2^T$.

We can obtain well-known results for conventional doubly coprime factorization and Youla parameterization of stabilizing controllers in many references (Vidyasagar, 1985; Green, 1995). Let the subscript "ur" represent "unconstrained right coprime factor" and "ul" "unconstrained left coprime factor" to represent conventional doubly coprime factorization.

**Remark 1** For the original plant:

$$ G(s) = \begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix} = M_{ul}^{-1} N_{ul} = N_{ur} M_{ur}^{-1}. $$

(6)

The doubly coprime factorization is given by

$$ \begin{bmatrix} Y_{ur} & X_{ur} \\ -N_{ul} & M_{ul} \end{bmatrix} \begin{bmatrix} M_{ur} & -X_{ul} \\ N_{ur} & Y_{ul} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} $$

(7)

with each transfer functions in $\mathcal{RH}_\infty$, and where

$$ \begin{bmatrix} Y_{ur} & X_{ur} \\ -N_{ul} & M_{ul} \end{bmatrix} = \begin{bmatrix} A + H_2 C_2 & B_2 & -H_2 \\ -F_2 & I & 0 \\ C_2 & 0 & I \end{bmatrix} $$

(8)

and

$$ \begin{bmatrix} M_{ur} & -X_{ul} \\ N_{ur} & Y_{ul} \end{bmatrix} = \begin{bmatrix} A + B_2 F_2 & B_2 & -H_2 \\ F_2 & I & 0 \\ C_2 & 0 & I \end{bmatrix} $$

(9)

These are elementary results for doubly coprime factorization.

Above results can be independently derived from internal stability theory. Also, if $H_2$ Riccati solutions $X_2, Y_2$ are used in $F_2, L_2$, then coprime factors can be normalized such as $\|N_{ul} \ M_{ul}\|_\infty = 1$ and $\|M_{ur} \ N_{ur}\|_\infty = 1$ for all frequencies.

**Remark 2** All stabilizing controllers are parameterized from the $H_2$ central controller $X \ Y^{-1} = X \ Y^{-1} X^{-1} X^{-1}$ as follows:

$$ K = -(X_{ul} - N_{ul} Q)(Y_{ul} + M_{ur} Q)^{-1} $$

(10)

in which

$$ J_u = \begin{bmatrix} -Y_{ul}^{-1} X_{ur} & Y_{ul}^{-1} \\ Y_{ul}^{-1} & -Y_{ul}^{-1} N_{ur} \end{bmatrix} $$

(11)
If the doubly coprime factorization takes the form given in (8) and (9), then
\[
J_\nu = \begin{bmatrix}
A + B_2 F_2 + H_2 C_2 & -H_2 & B_2 \\
F_2 & 0 & 1 \\
-C_2 & 1 & 0
\end{bmatrix}
\]  
(12)
where \( Q \) should be stable.

Main result of Remark 2 describes typical standard \( \mathcal{H}_\infty \) state space solution.

4 MAIN RESULTS: Constrained Youla Parameterization

When the central controller is given, to parameterize all stabilizing controllers by doubly coprime factorization, first of all, we need to build artificial plant which can be derived from the central controller and \( \mathcal{H}_\infty \) norm constraint. This will be called as the "constrained plant". By using constrained plant and central controller, doubly coprime factors can be made and Youla parameterization can be achieved by using the constrained doubly coprime factorization.

4.1 Constrained Plant

Definition 1 Consider the system of (3) satisfying above five assumptions. Inequality constraint defined for any positive constant \( \gamma > \gamma_{\text{opt}} \)
\[
\|z\|_2^2 \leq \gamma^2 \|w\|_2^2
\]  
(13)
is said to be \( \mathcal{H}_\infty \) norm constraint, and the plant induced from \( \mathcal{H}_\infty \) norm constraint is said to be constrained plant.

This definition is very meaningful in view of the disturbance attenuation, and it gives us to the inequality constraint for \( \infty \)-norm of closed-loop transfer matrix as shown in Fig. 4.1, namely,

\[
\|F_\ell(P, K)\|_\infty \leq \gamma.
\]

Figure 1: \( \mathcal{H}_\infty \) norm constraint

Proposition 1 Consider a linear system of (3) for which assumptions 1-5 hold. Suppose the matrix \( X_\infty \) of (4) exists, then the closed loop dynamics is expressed by
\[
x = (A + \gamma^2 B_1 B_1^T X_\infty - B_2 B_2^T X_\infty) x
\]  
(14)
and the matrix \( A + \gamma^2 B_1 B_1^T X_\infty - B_2 B_2^T X_\infty \) has all its eigenvalues in the open left half of the complex plane. The worst case disturbance input is

\[
w = \gamma^{-2} B_1^T X_\infty x
\]  
(15)
and the optimal control input is
\[
u = F_\infty x = -B_1^T X_\infty x.
\]  
(16)
The closed-loop dynamics of (14) is the dynamics constrained by \( \mathcal{H}_\infty \) norm constraint of (13).

If the solution of Riccati equation \( X_\infty \) exists, then the closed-loop dynamics of (14) is asymptotically stable since the riccati operator of (4) gives the stable invariant subspace by the property of riccati operator. If we assume that the symmetric and semi-positive definite matrix \( X_\infty \) exists, we can differentiate Lyapunov candidate of \( x^T X_\infty x \):
\[
\frac{d}{dt}(x^T X_\infty x) = x^T (A^T X_\infty X_\infty + X_\infty A) x + w^T B_1^T X_\infty x
\]
\[
+ z^T X_\infty B_1 w + u^T B_2^T X_\infty x + z^T X_\infty B_2 u
\]
\[
= -\|z\|_2^2 + \gamma^2 \|w\|_2^2 + \|u + B_2^T X_\infty x\|_2^2
\]
\[
- \gamma^2 \|w - \gamma^{-2} B_1^T X_\infty x\|_2^2 < 0
\]  
(17)
where the Riccati equation for \( X_\infty \) is
\[
A^T X_\infty X_\infty + X_\infty A + C_1^T C_1
\]
\[
+ \gamma^{-2} X_\infty B_1 B_1^T X_\infty - X_\infty B_2 B_2^T X_\infty = 0.
\]  
(18)
Hence, the \( \mathcal{H}_\infty \) norm constraint of (13) can be interpreted as (17). Note that \( w := \gamma^{-2} B_1^T X_\infty x \) is the worst case disturbance input in the sense that it maximizes the quantity \( \|z\|_2^2 - \gamma^2 \|w\|_2^2 \) in Definition 1 for the minimizing value of \( u = -B_2^T X_\infty x \); that is the \( u \) making \( u + B_2^T X_\infty x = 0 \) and \( w \) making \( w - \gamma^{-2} B_1^T X_\infty x = 0 \) is value satisfying a saddle point condition (Doyle, 1989). \( \mathcal{H}_\infty \) norm constraint gives the functional relations of (15) and (16) between stabilizable maximal disturbance, control input and state vector.

Proposition 2 Consider the closed-loop dynamics of (14) in Proposition 1. Suppose the \( X_\infty, Y_\infty \) and \( Z_\infty \) exist, then the constrained plant in Definition 1 is expressed by
\[
P_s = \begin{bmatrix}
A_s & Z_\infty B_2 \\
C_2 & 0
\end{bmatrix}
\]  
(19)
where
\[
A_s = A + \frac{1}{\gamma^2} B_1 B_1^T X_\infty + \frac{1}{\gamma^2} Z_\infty Y_\infty X_\infty B_2 B_2^T X_\infty
\]
and \( P_s \) denotes the constrained plant. The effect on system matrix caused by worst-case disturbance input is
\[
\frac{1}{\gamma^2} B_1 B_1^T X_\infty
\]  
(20)
and the effect on system matrix caused by worst-case state estimation is
\[
\frac{1}{\gamma^2} Z_\infty Y_\infty X_\infty B_2 B_2^T X_\infty
\]  
(21)
Now, it can be seen that the constrained plant includes the effect of worst-case disturbance and worst-case state estimation.

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In the problem of full state feedback, since the measure of state vector is available, \( Y_{\infty} = 0 \) and \( Z_{\infty} = I \). However, without loss of generality, the constrained plant can be shown as follows by a simple algebraic calculation for (14):

\[
\dot{x} = (A + \gamma^{-2}B_1B_1^TX_{\infty} - Z_{\infty}Z_{\infty}^{-1}B_2B_2^TX_{\infty})x
= (A + \gamma^{-2}B_1B_1^TX_{\infty} + \gamma^{-2}Z_{\infty}Y_{\infty}X_{\infty}B_2B_2^TX_{\infty})x + Z_{\infty}B_2u
\]

where \( Z_{\infty} \) is real positive definite matrix.

However, in the problem of output feedback, can we say that the constrained plant is expressed like (19)? The answer is “positive”. That is the reason why there is no interference on the stability between observer and system dynamics by the separation principle. Therefore, let us investigate the stability for the state error dynamics. Consider \( H_{\infty} \) central controller \((K_s)\) as follows:

\[
\dot{x} = Ax + B_1\dot{w} + B_2u + Z_{\infty}H_{\infty}(C_2\dot{x} - y)
\]

\[
u = F_{\infty}\dot{x}	ext{ and } \dot{w} = \gamma^{-2}B_1^TX_{\infty}\dot{x}
\]

or equivalently

\[
K_s = \begin{bmatrix}
A_c \\
F_{\infty}
\end{bmatrix} - \begin{bmatrix}
-Z_{\infty}H_{\infty} \\
0
\end{bmatrix}
\]

where \( A_c = A + B_2F_{\infty} + \gamma^{-2}B_1B_1^TX_{\infty} + Z_{\infty}H_{\infty}C_2 \). Define the state error vector as \( e = x - \dot{x} \) and consider the state error dynamics to obtain worst-case state estimation from the stability of state error dynamics which is obtained by subtracting (23) from (3):

\[
\dot{e} = (A + \gamma^{-2}B_1B_1^TX_{\infty} + Z_{\infty}H_{\infty}C_2)e.
\]

Differentiate the Lyapunov candidate of \( \frac{1}{2}e^TZ^{-1}e \) for the state error dynamics becomes

\[
\frac{d}{dt}(\frac{1}{2}e^TZ^{-1}e) = \frac{1}{2}e^TZ^{-1}(A + \gamma^{-2}B_1B_1^TX_{\infty} + Z_{\infty}H_{\infty}C_2)Z^{-1}
+ Z_{\infty}^{-1}(A + \gamma^{-2}B_1B_1^TX_{\infty} + Z_{\infty}H_{\infty}C_2)e
= \frac{1}{2}e^TZ_{\infty}B_2B_2^TX_{\infty}Z_{\infty}^{-1}e - \frac{1}{2}e^T\gamma^{-2}C_2^T\gamma^{-2}C_2e
\]

where the Riccati equation for \( Z \) is

\[
Z(A + \gamma^{-2}B_1B_1^TX_{\infty})^T + (A + \gamma^{-2}B_1B_1^TX_{\infty})Z + B_2B_2^T - ZC_2^TC_2Z
+ \gamma^{-2}Z_{\infty}B_2B_2^TX_{\infty}Z_{\infty} = 0.
\]

(27)

Therefore, the state error dynamics of (25) is asymptotically stable.

The Riccati equation of (27) is obtained by the similarity transformation of Hamiltonian matrix associated with \( Y_{\infty} \) of (5). Introducing the transformation matrix

\[
T = \begin{bmatrix}
I & -\gamma^{-2}X_{\infty} \\
0 & I
\end{bmatrix},
\]

then the stable invariant subspace of Riccati equation for \( Z \) is given by

\[
T\begin{bmatrix}
I \\
Y_{\infty}
\end{bmatrix} = \begin{bmatrix}
I - \gamma^{-2}X_{\infty}Y_{\infty} \\
Y_{\infty}
\end{bmatrix}.
\]

(29)

The stable solution of Riccati equation of (27) is \( Y_{\infty}(I - \gamma^{-2}X_{\infty}Y_{\infty})^{-1} \). We can know that \( Z = Y_{\infty}(I - \gamma^{-2}X_{\infty}Y_{\infty})^{-1} = (I - \gamma^{-2}Y_{\infty}X_{\infty})^{-1}Y_{\infty} = Z_{\infty}Y_{\infty} \). The stable error dynamics of riccati (27) is expressed by

\[
\dot{e} = (A + \gamma^{-2}B_1B_1^TX_{\infty} + \gamma^{-2}Z_{\infty}Y_{\infty}X_{\infty}B_2B_2^TX_{\infty} + Z_{\infty}H_{\infty}C_2)e,
\]

and this is error dynamics which can be stabilized by central controller. The worst-case state estimation can be found from the difference between the real error dynamics of (25) and error dynamics which can be stabilized like (30). That is, the worst-case state estimation is \( \frac{1}{2}e^TZ_{\infty}Y_{\infty}X_{\infty}B_2B_2^TX_{\infty} \): that is the stabilizable maximum perturbation for state error dynamics. As shown in Fig. 4.1, the system matrix of the constrained plant includes the bad-effect which can be caused by worst-case disturbance \((\frac{1}{2}B_1B_1^TX_{\infty})\) and worst-case state estimation \((\frac{1}{2}Z_{\infty}Y_{\infty}X_{\infty}B_2B_2^TX_{\infty})\). With the mathematical expression, the system matrix of constrained plant (19) includes implicitly Riccati solutions dependent upon \( \gamma \). If \( \gamma \rightarrow \infty \), then we can obtain original plant of \( C_2(I-A)^{-1}B_2 \) from the constrained plant of (19).

4.2 Constrained Doubly Coprime Factorization

In this section, we show the doubly coprime factorization for the constrained plant \( (P_s) \) and central controller \((K_s) \). Let the subscript “r” represent “constrained right coprime factor” and “l” “constrained left coprime factor”.

Theorem 1 For a given generalized plant (3) satisfying five assumptions, if \( X_{\infty} \), \( Y_{\infty} \) and \( Z_{\infty} \) exist, then we can find the constrained plant \( (P_s) \) (19) by \( H_{\infty} \) central controller \((K_s) \) of (24). Let us factorize \( P_s \) as \( N_lM^{-1}_l = M^{-1}_rN_l \) and \( K_s \) as \( -X_sY^{-1}_l = -Y^{-1}_lX_s \) satisfying Bezout Identity as follows:

\[
\begin{bmatrix}
Y_r & X_r \n
N_l & M_l
\end{bmatrix}
\begin{bmatrix}
M_r & -X_l \n
-N_l & Y_l
\end{bmatrix}
= \begin{bmatrix}
I & 0 

0 & I
\end{bmatrix}
\]

(31)
with each transfer matrices in $\mathcal{R}\mathcal{H}_\infty$, and where

$$
\begin{bmatrix}
Y_r & X_r \\
-N_r & M_r
\end{bmatrix} =
\begin{bmatrix}
A_r & Z_\infty B_2 & -Z_\infty H_\infty \\
-F_\infty & I & 0 \\
-C_2 & 0 & I
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
M_r & -X_l \\
N_r & Y_l
\end{bmatrix} =
\begin{bmatrix}
A_l & Z_\infty B_2 & -Z_\infty H_\infty \\
F_\infty & I & 0 \\
C_2 & 0 & I
\end{bmatrix}
$$

where $A_r = A + \frac{1}{\gamma} B_1 B_1^T X_\infty + \frac{1}{\gamma^2} Z_\infty Y_\infty X_\infty B_2 B_1^T X_\infty + Z_\infty H_\infty C_2$ and $A_l = A + \frac{1}{\gamma} B_1 B_1^T X_\infty + B_2 F_\infty$.

The proof is given in Appendix A. If $X_\infty$, $Y_\infty$ and $Z_\infty$ exist, the left and right constrained coprime factors can be found in any cases.

4.3 All Stabilizing Controller

All stabilizing controllers are parameterized with free parameter $Q$.

**Theorem 2** All stabilizing controller is described as follows

$$
K = -(X_l - N_r Q)(Y_l + M_r Q)^{-1}
= \mathcal{F}_r(J, Q)
$$

in which

$$
J = \begin{bmatrix}
-Y_l^{-1} X_r & Y_l^{-1} \\
Y_l^{-1} & -Y_l^{-1} N_r
\end{bmatrix}
$$

If the doubly coprime factorization takes the form given in (32) and (33) of Theorem 1, then

$$
J = \begin{bmatrix}
A_c & Z_\infty H_\infty & Z_\infty B_2 \\
F_\infty & 0 & I \\
-C_2 & 0 & I
\end{bmatrix}
$$

where $A_c = A + B_2 F_\infty + \gamma^{-2} B_1 B_1^T X_\infty + Z_\infty H_\infty C_2$.

The proof is omitted because it consists of only algebraic manipulations.

(36) is the standard state-space $\mathcal{H}_\infty$ solution suggested by Doyle et al. (1989). This shows that the standard state-space $\mathcal{H}_\infty$ solution with a free parameter can also be obtained using constrained doubly coprime factorization. If we consider the point that central controller can be obtained by various methods, then we can parameterize all stabilizing $\mathcal{H}_\infty$ controllers with free parameter $Q$ using the proposed doubly coprime factorization.

4.4 Closed-loop Transfer Matrix

The closed-loop transfer matrix consists of a generalized plant and controller. Therefore, the internal stability between the generalized plant and all stabilizing controllers parameterized by the constrained plant and central controller should be assured. Bezout Identity between a generalized plant and a controller is not satisfied because the generalized plant is not constrained. However, we know that $K$ is an internally stabilizing controller for $P$ if and only if an internally stabilizing controller for $P_{22}$ (Green, 1995).

**Lemma 1** Suppose $P_{22} = M^{-1} M_{ul} N_{ul} = N_{ur} M^{-1}$ and $K = -Y_l^{-1} X_r = -X_l Y_l^{-1}$. If the feedback system is internally stable for $P_{22}$, then

1. $X_r N_{ul} + Y_r M_{ur}$ is invertible in $\mathcal{R}\mathcal{H}_\infty$.
2. $N_{ul} X_l + M_{ul} Y_l$ is invertible in $\mathcal{R}\mathcal{H}_\infty$.

The proof is given in Appendix B. Even when $Y_r$, $X_r$, $X_l$, and $Y_l$ are replaced by $Y_r + Q N_l$, $X_r - Q M_l$, $X_l - M_r Q$ and $Y_l + N_r Q$, respectively, Lemma 1 can be satisfied.

Now, consider the closed-loop transfer matrix of

$$
\mathcal{F}_l(P, K) = P_{11} + P_{12} K(I - P_{22} K)^{-1} P_{21}
$$

Let $P_{22} = M^{-1} M_{ul} N_{ul}$ and $K = -(X_l - M_r Q)(Y_l + N_r Q)^{-1}$, then

$$
\mathcal{F}_l(P, K) = P_{11} + P_{12} (M_r Q - X_l) U^{-1} M_{ul} P_{21}
$$

where $U = M_{ul} Y_l + N_{ul} X_l + (M_r N_r - N_{ul} M_r) Q$. The matrix $U$ is not identity matrix because the plant $P_{22}$ is not constrained plant $(M_{ul} N_l)$. However, $U \in \mathcal{R}\mathcal{H}_\infty$ and $U^{-1} \in \mathcal{R}\mathcal{H}_\infty$ by Lemma 1. For the given plant of $P_{22} = M^{-1} M_{ul} N_{ul}$, we can find $X$ and $Y$ satisfying

$$
N_{ul} X + M_{ul} Y = U
$$

where $U, U^{-1} \in \mathcal{R}\mathcal{H}_\infty$. There are many $X$ and $Y$ which satisfy the relation of (39). Especially, when $U = I$, the relation of (39) is said to be "Bezout Identity", then a set of solutions $X$ and $Y$ is called a particular solution.

**Remark 3** Consider the relation of (39), if $U, U^{-1} \in \mathcal{R}\mathcal{H}_\infty$, then

$$
X = X_l - N_r Q \quad \text{and} \quad Y = Y_l + M_r Q
$$

can be a solution selected by choosing constant $\gamma$ among many solutions for (39), and if $U = I$,

$$
X = X_{ul} - N_{ur} Q \quad \text{and} \quad Y = Y_{ul} + M_{ur} Q
$$

is a particular solution, where $Q$ is a free stable parameter.

$\mathcal{H}_\infty$ controller consists of $-XY^{-1} = -(X_l - N_r Q)(Y_l + M_r Q)^{-1}$, while $\mathcal{H}_2$ controller consists of $-XY^{-1} = -(X_{ul} - N_{ur} Q)(Y_{ul} + M_{ur} Q)^{-1}$. If right coprime factors for plant $P_{22} = N_{ur} M^{-1}$ is used, we can obtain another description for controllers $(Y^{-1} X)$. As $\gamma$ is relaxed to $\infty$, since $X_l \to X_{ul}$, $Y_l \to Y_{ul}$, $M_r \to M_{ur}$ and $N_r \to N_{ur}$, therefore $U \to I$.

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5 CONCLUDING REMARKS

Youla parameterization was suggested in which $H_{\infty}$ norm of closed-loop transfer matrix could be smaller than any prescribed value by using the constrained doubly coprime factorization. The parameterization of all $H_{\infty}$ controller was obtained from the given central controller and the concept of constrained plant. This means that if the central controller is given, we can parameterize all $H_{\infty}$ controller with a free parameter by proposed method.

REFERENCES


A Proof of Theorem 1

The doubly coprime factorization for constrained plant by $\|F_I(P, K)\|_{\infty} \leq \gamma$ is as follows. By proposition 2, if $X_{\infty}, Y_{\infty}$ and $Z_{\infty}$ exist, we can always find the constrained plant of (19) from given generalized plant of (3). From Riccati equation of (4), we readily know that the matrix

$$A_1 = A + \frac{1}{2} B_1 B_1^T X_{\infty} - B_1 B_2^T X_{\infty}$$

is negative definite (namely, all eigenvalues are in open left half plane) by the property of Riccati operator. Also, from the Riccati equation of (27), we know that the matrix

$$A_r = A + \frac{1}{2} B_r B_r^T X_{\infty} + \frac{1}{2} Z_{\infty} Y_{\infty} X_{\infty} B_2 B_2^T X_{\infty} - Z_{\infty} Y_{\infty} C_r^T C_2$$

is negative definite. Therefore, transfer matrices defined by (32) and (33) are both in $\mathcal{RH}_{\infty}$. Suppose that $P_r = N_r M_r^{-1}$ for plant $P_r$ is a right coprime factorization and that $X_r, Y_r \in \mathcal{RH}_{\infty}$ satisfy $X_r N_r + X_r M_r = I$. Suppose also that $P_r = N_r M_r^{-1}$ is left coprime factorization and that $X_r Y_r \in \mathcal{RH}_{\infty}$ satisfy $N_r X_r + M_r Y_r = I$.

Then

$$\begin{bmatrix} Y_r & X_r \\ -N_r & M_r \\ M_r & M_r R - X_r \\ -N_r & N_r R + Y_r \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

in which $R = Y_r X_r - X_r Y_r$. This shows that for any right and left coprime factorizations of same transfer function matrix (i.e. $N_r M_r^{-1} = M_r^{-1} N_r$), there exist transfer function matrices $X_r, Y_r, X_l$ and $Y_l$ all in $\mathcal{RH}_{\infty}$, such that the generalized Bezout equation:

$$\begin{bmatrix} Y_r & X_r \\ -N_r & M_r \\ M_r & M_r R - X_r \\ -N_r & N_r R + Y_r \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is satisfied. This is known as a doubly coprime factorization, and we shall use it to parametrize all stabilizing controllers. The proof of (32) and (33) follows from some algebraic manipulations.

The verification of the identities $P_r = N_r M_r^{-1} = M_r^{-1} N_r$ is as follows. In calculation of $M_r^{-1}$, use $Z_{\infty} B_2 F_{\infty} = \frac{1}{2} Z_{\infty} Y_{\infty} X_{\infty} B_2 F_{\infty} + B_2 F_{\infty}$.

$$N_r M_r^{-1} = \begin{bmatrix} A_r & Z_{\infty} B_2 F_{\infty} \\ 0 & A_r - Z_{\infty} B_2 F_{\infty} \end{bmatrix} \begin{bmatrix} Z_{\infty} B_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(44)

If a similarity transformation matrix $T = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$, $T^{-1} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$ is applied to (44), then $P_r$ is obtained. Also, by similar procedure to above, $M_r^{-1} N_r = P_r$. Hence, we conclude that

$$P_r = N_r M_r^{-1} = M_r^{-1} N_r.$$ 

(45)

Matrix $R$ of (42) must be 0 if and only if $M_r N_r = M_r N_r$. It implies $X_r Y_r^{-1} = Y_r^{-1} X_r$. For left and right coprime factors of the constrained plant, the controller $K$ can be composed of $-X_r Y_r^{-1}$ and $-Y_r^{-1} X_r$, and it is said to be a central controller in view of parametrization of stabilizing controllers.

B Proof of Lemma 1

Internal stability theory is equivalent to

$$\begin{bmatrix} I & -K \\ -P_{22} & I \end{bmatrix} \in \mathcal{RH}_{\infty}.$$ 

(46)

Let us show item 1;

$$\begin{bmatrix} I & -K \\ -P_{22} & I \end{bmatrix} = \begin{bmatrix} I & Y_r^{-1} X_r \\ M_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & Y_r^{-1} X_r \\ 0 & 0 \end{bmatrix} \in \mathcal{RH}_{\infty}.$$ 

Therefore, for

$$\begin{bmatrix} Y_r M_r & X_r \\ -N_r & -N_r \end{bmatrix} \in \mathcal{RH}_{\infty},$$

$X_r N_r + Y_r M_r$ should be invertible in $\mathcal{RH}_{\infty}$.

The proof for item 2 is similar to the above.