Singularity-Robust Inverse Kinematics Using Lagrange Multiplier for Redundant Manipulators

Youngjin Choi
Division of Electrical Engineering and Computer Science,
Hanyang University,
Ansan 426-791, Republic of Korea
e-mail: cyj@hanyang.ac.kr

1 Introduction

Redundant manipulators have more degrees of freedom than those required to execute a given task [1,2]. The remaining degrees of freedom are referred to as redundancy. The redundant manipulators can execute another task using the remaining redundancy, e.g., obstacle avoidance, minimization of joint torque, and maximization of dexterity measures [3,4]. In general, if a task does not come into conflict with other tasks, then redundant manipulators can simultaneously accomplish several tasks by making use of redundancy. However, if there are conflicts between a task and other tasks, less important tasks should be given up according to the order of priority of tasks. The concept of task-priority inverse kinematics for a redundant manipulator was proposed for the first time in Refs. [5,6]. According to the order of priority, a task with higher priority is firstly performed and a task with lower priority is performed utilizing the redundancy on task with higher priority [7]. However, this algorithm suffers from kinematic and algorithmic singularities [8]. In order to alleviate the bad behavior of either inverse or pseudoinverse of a Jacobian matrix, which is derived from the differential kinematics equation of a redundant manipulator, the damped least-squares inverse was developed in Ref. [9]. The damped least-squares inverse of the Jacobian matrix has the following form:

\[ J^* = J^T (JJ^T + \epsilon I)^{-1} \] (1)

where \( \epsilon \) means the positive constant dampening factor. It is essential to select a suitable value for the dampening factor; small values of \( \epsilon \) give accurate solutions but low robustness in the neighborhood of singular configurations, while large values of \( \epsilon \) result in low tracking accuracy even though feasible and accurate solutions would be possible [4].

In this paper, the inverse kinematics problem will be formulated to be the minimization problem composed of a least-squares con-

dition of secondary task error and a minimum norm condition of configuration velocity, subject to an equality constraint of primary task. Second, in the procedure of minimization for a given objective function, a new inverse kinematics algorithm is derived. Third, since nonzero Lagrange multiplier values appear in the neighborhood of a singular configuration of a robotic manipulator, we choose them as a natural choice of the dampening factor to alleviate the ill-conditioning of matrix inversion, ultimately for singularity-robust inverse kinematics. Finally, the effectiveness of the suggested singularity-robust inverse kinematics is shown through a numerical simulation about deburring and conveyance tasks of a dual arm manipulator system.

[DOI: 10.1115/1.2957632]

2 Optimization for Singularity-Robust Inverse Kinematics

There are two ways to solve an optimization problem subject to an equality constraint; one is an instantaneous (local) optimization using the Jacobian relation of the given nonlinear function, and another is a global optimization of nonlinear function. If enough computational time is allowed to solve the optimization problem, then the global optimization is preferred to the local scheme. However, since the robotic manipulator requires a real-time control, the local optimization has been used in view of real-time implementation. In the following sections, the local optimization will be used to obtain a singularity-robust inverse kinematics.

2.1 Problem Statement. Let us focus our attention on two tasks with the priority order in the following form:

\[ r_i = k_i(q) \quad \text{for } i = 1, 2 \] (2)

where \( r_1 \in \mathbb{R}^{m_1} \) is an \( m_1 \)-dimensional primary task function with higher priority, \( r_2 \in \mathbb{R}^{m_2} \) is an \( m_2 \)-dimensional secondary task function with lower priority, \( q \in \mathbb{R}^n \) is an \( n \)-dimensional joint configuration vector, and \( k_i(\cdot) \) is a nonlinear direct kinematics equation of a redundant manipulator for a given \( i \)-th task. Also, the given two tasks can be differentiated with time and then defined by using Jacobian matrices as follows:

Contributed by the Dynamic Systems, Measurement, and Control Division of ASME for publication in the JOURNAL OF DYNAMIC SYSTEMS, MEASUREMENT, AND CONTROL. Manuscript received September 28, 2007; final manuscript received May 12, 2008; published online August 4, 2008. Assoc. Editor: Yossi Chait.
\[ \mathbf{r}_i = J_i(q) \mathbf{q} \quad \text{for } i = 1, 2 \]  
where \( J_i(q) \approx \frac{\partial \mathbf{r}_i(q)}{\partial \mathbf{q}}^T \in \mathbb{R}^{m_i \times n} \). In addition, \( J_i(q) \) means a primary task Jacobian matrix and \( J_i(q) \) a secondary task Jacobian matrix.

Now, let us define the objective function composed of the sum of least-squares secondary task error condition and minimum norm condition subject to the primary task execution condition in the following form:

minimize \[ \| \mathbf{r}_2 - J_2(q) \mathbf{q} \|^2 + \varepsilon^2 \| \mathbf{q} \|^2 \]
subject to \[ \mathbf{r}_1 = J_1(q) \mathbf{q} \]

where \( \varepsilon \) is referred to as a dampening factor. Here, we assume that the redundant manipulator has degrees of freedom more than or similar to those required to execute two tasks, namely, \( n > m_1 + m_2 \). In addition, we assume that the primary task Jacobian has a full rank for primary task execution. The existence condition of the solution of the above objective function will be suggested in the following section.

### 2.2 Existence Condition

To solve an optimization problem of objective function (4), first of all, the Lagrangean function should be defined as follows:

\[ \mathcal{L}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{q}, \lambda) = \frac{1}{2} (\mathbf{r}_2 - J_2(q) \mathbf{q})^T \mathbf{r}_2 + \frac{1}{2} \varepsilon^2 \mathbf{q}^T \mathbf{q} + \lambda^T (\mathbf{r}_1 - J_1(q) \mathbf{q}) \]

where \( \lambda \in \mathbb{R}^{m_1} \) means a Lagrange multiplier vector. Then, the first-order derivatives can be obtained to find the local minimum as follows:

\[ \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = J_2^T \mathbf{r}_2 + \varepsilon^2 \mathbf{q} - J_1^T \lambda = 0 \]

\[ \frac{\partial \mathcal{L}}{\partial \lambda} = \mathbf{r}_1 - J_1(q) \mathbf{q} = 0 \]

Now, rearranging the above equations gives the following compact equation:

\[
\begin{bmatrix}
J_2^T + \varepsilon^2 I \\
J_1^T \\
0 \\
- \lambda
\end{bmatrix}
\begin{bmatrix}
\mathbf{q} \\
\mathbf{r}_2
\end{bmatrix}
= 
\begin{bmatrix}
J_2^T \mathbf{r}_2 \\
\mathbf{r}_1
\end{bmatrix}
\]

where \( I \) and \( 0 \) are identity and zero matrices with suitable dimensions, respectively.

As a result, we can get the following optimum solution and Lagrange multiplier by using the inverse (Eq. (11)) for Eq. (6):

\[
\mathbf{q}_o = W^{-1} J_2 \mathbf{r}_2 + \varepsilon^2 I \mathbf{q} + (I - W^{-1} J_2 Y^{-1} J_1 W^{-1}) W^{-1} J_2 \mathbf{r}_2
\]

(12)

\[
\lambda = (Y^{-1} - I) \mathbf{r}_1 - Y^{-1} J_1 W^{-1} J_2 \mathbf{r}_2
\]

(13)

Also, since the existence condition of Eq. (7) is equivalent to

\[ J_2^T \mathbf{r}_2 \in \mathcal{R}(W) \quad \text{and} \quad \mathbf{r}_1 \in \mathcal{R}(Y) \]

if and only if condition (14) is satisfied, the optimal solution of inverse kinematics exists for two tasks with priority. In addition, the optimal solution of Eq. (12) can be rearranged using the concept of weighted pseudoinverse as follows:

\[
\mathbf{q}_o = J_2^T \mathbf{r}_2 + (I - W^{-1} J_2 Y^{-1} J_1 W^{-1}) W^{-1} J_2 \mathbf{r}_2
\]

(15)

where \( W = W^{-1} J_2^T (J_2 J_2^T)^{-1} \). Now, we need to understand the role of the weighting matrix \( W \) in the optimal solution. For a comparative study, let us assign the weighting matrix to be an identity matrix; then we can get the following equation:

\[
\mathbf{q}_o = J_2^T \mathbf{r}_2 + (I - J_2 Y^{-1} J_1^T)^{-1}
\]

(16)

where \( J_2 = J_2 (J_2 J_2^T)^{-1} \). Since the comparison of Eqs. (15) and (16) helps in understanding the role of the weighting matrix, the conceptual diagram about the role of the weighting matrix is shown in Fig. 1. Firstly, in Fig. 1, we can get the following property:
**Property 1.** If the weighting matrix $W$ has a full rank, then the following statements are satisfied in Fig. 1.

1. $\mathcal{R}(J_1^{W*})$ and $\mathcal{N}(J_1)$ are $W$-orthogonal.
2. $\mathcal{R}(J_1^T)$ and $\mathcal{N}(J_1)$ are geometrically orthogonal, where $\mathcal{R}(\cdot)$ and $\mathcal{N}(\cdot)$ means the null space.

**Proof.** Item 1: since item 1 implies the orthogonality in the weighted metric, we show that the next relation is satisfied,

$$(J_1^{W*})^TW(I - J_1^TJ_1; J_1^{W*}) = (J_1W^{-1}J_1^TW(I - J_1^TJ_1; J_1^{W*}) = 0$$

Item 2: we can easily show that $\mathcal{R}(J_1^T)$ and $\mathcal{N}(J_1)$ are geometrically orthogonal because

$$J_1(I - J_1^TJ_1) = 0$$

As shown in Fig. 1 and Property 1, Eq. (16) is composed of the sum of the primary task component $J_1^W r_1$ embedded on $\mathcal{R}(J_1)$ and the secondary task component $(I - J_1^TJ_1)J_1^W r_2$ embedded on $\mathcal{N}(J_1)$, which is geometrically orthogonal to $\mathcal{R}(J_1)$. On the other hand, the optimal solution of Eq. (15) is composed of the sum of the primary task component $J_1^W r_1$ embedded on $\mathcal{R}(J_1)$ and the secondary task component $(I - J_1^WJ_1)W^{-1}J_1^W r_2$ embedded on $\mathcal{N}(J_1)$, which is not geometrically orthogonal to $\mathcal{R}(J_1)$. The weighting matrix $W$ is able to rotate the embedded range space for primary task execution with a suitable rotation angle; in other words, it changes the orthogonal null space projection method into the efficient null space projection method for secondary task execution. Despite the role of the weighting matrix, the secondary task error may remain with a nonzero $\epsilon$; in other words, the primary task error $e_1(J_1^W r_1 - J_1 q_1)$ is zero, but the secondary task error $e_2(J_1^W r_2 - J_1 q_2)$ is, in general, not equal to zero. Though the optimal solution (Eq. (15)) must be robust in the neighborhood of singularities, it may have the secondary task error. However, the secondary task error can be reduced when $\epsilon=0$ in the objective function (Eq. (4)). This will be illustrated in the following section.

**2.4 Optimal Solution When $\epsilon=0$.** Now, let us reconsider the meaning of the objective function (Eq. (4)). If the minimum norm condition is removed in the objective function, namely, $\epsilon=0$, then the solution of minimization of secondary task error can easily be obtained by letting $\epsilon=0$ and changing the inverse into the Moore–Penrose pseudoinverse in Eqs. (5)–(9) as follows:

$$Z^* = \begin{bmatrix} W^* - W^*J_1^T Y^* & W^*J_1^T Y^* \\ Y^*J_1W^* & YY^* - Y^* \end{bmatrix}$$

(17)

Then, the optimal solution of the secondary task error minimization problem has the following form:

$$q = J_1^W r_1 + (I - J_1^WJ_1)W^*J_1^W r_2$$

(18)

$$\lambda = (Y^* - YY^*)r_1 - Y^*J_1W^*J_1^W r_2$$

(19)

where $J_1^{W*} = W^*J_1^T Y^* = W^*J_1^T (J_1W^*J_1)^{-1}$; in this case, we can get another property as follows:

**Property 2.** If there is no intersection between $\mathcal{R}(J_1^T)$ and $\mathcal{R}(J_2^T)$, then the following statements are satisfied in Fig. 1.

1. $\mathcal{R}(J_1^T)$ and $\mathcal{R}(J_2^T)$ are $W^*$-orthogonal.
2. $\mathcal{R}(J_1^{W*})$ and $\mathcal{R}(J_2^T)$ are geometrically orthogonal.

**Proof.** Item 1: At first, we show that $J_1^W J_1^T$ is idempotent. Since the same is true for the symmetric matrix $W^*J_1J_1^T$, we show that the eigenvalues of $W^*J_1J_1^T$ are either 1 or zero. Let $\kappa$ be the eigenvalues of $W^*J_1J_1^T$, and $x$ the corresponding eigenvector; then we have the next relation,

$$W^*J_1J_1^T x = \kappa x$$

$$W^*W^*J_1J_1^T x = \kappa W x$$

$$J_1^T J_1 x = \kappa (J_1^T J_1 + J_2 J_2^T) x$$

Therefore,

$$(1 - \kappa) J_1^T J_1 x = \kappa J_1^T J_1 x$$

Since there is no intersection between the range space of $J_1^T$ and the range space of $J_2^T$, the above equation implies that
Thus, $J J^T x = 0$ implies $\kappa = 0$ and $J J^T x \neq 0$ implies $\kappa = 1$. Hence the eigenvalues are either one or zero. Using the idempotent property of $J W J^T$, we can show that $R(J^T)$ and $R(J^T)$ are $W^T$-orthogonal as follows:


Hence,

$$J W J^T = 0$$

(20)

Item 2: We can easily show that $R(J W)$ and $R(J^T)$ are $W^T$-orthogonal using Eq. (20) as follows:

$$(J W)^T = (J W J^T)^T W J^T .$$

Through Property 2, we can see in Fig. 1 that $R(J W)$ and $R(J^T)$ are geometrically orthogonal to each other. However, when $\varepsilon = 0$, the weighting matrix $W$ may lose rank by intersecting $R(J^T)$ and $R(J^T)$. Actually, this intersection causes an algorithmic singularity referred to in Ref. [8]. When $\varepsilon = 0$, the optimal solution (Eq. (18)) of the secondary task error minimization problem may show the numerical instability in the neighborhood of singularities since the weighting matrix can lose rank. In addition, we can get the following matrix property by using $W^T$-orthogonality between $R(J^T)$ and $R(J^T)$:

$$J W J^T = J W J^T = 0$$

(21)

The difference between the optimal solution (Eq. (15)) for the objective function (Eq. (4)) and the optimal solution (Eq. (18)) when $\varepsilon = 0$ is whether the positive definiteness of the weighting matrix can be guaranteed or not. Actually, this means whether the suggested solution is robust or not in the neighborhood of singularities. Here, if there is no intersection between $R(J^T)$ and $R(J^T)$, the optimal solution (Eq. (18)) when $\varepsilon = 0$ has no secondary task error, but it may have difficulties in the neighborhood of singularities because $R(J^T)$ is able to intersect with $R(J^T)$ according to given tasks. On the contrary, the optimal solution (Eq. (15)) is robust in the neighborhood of singularities, but it may, in general, have the secondary task error as mentioned in the previous section. As a matter of fact, there is a trade-off between Eqs. (15) and (18); in the following section, we will suggest an alternative for this trade-off.

2.5 Singularity-Robust Inverse Kinematics. There are two kinds of major singularities in inverse kinematics. One is a kinematic singularity and the other is an algorithmic singularity. The kinematic singularity happens when the Jacobian matrix $J_1$ or $J_2$ loses rank. Also, the algorithmic singularity happens when the augmented Jacobian matrix $[J_1^T J_2^T]^T$ loses rank because the intersection of $R(J^T)$ and $R(J^T)$ causes losing rank. As mentioned in the previous section, the dampening factor $\varepsilon$ determines the trade-off between the exactness and the feasibility of the optimal solution. Fortunately, the Lagrange multiplier in Eq. (19) can be utilized as an alternative of trade-off. The Lagrange multiplier expresses how the optimal solution changes when the constraint equation is changed by a small amount [12]. In other words, the Lagrange multiplier represents the sensitivity of the optimal solution for constraint. The values of Lagrange multipliers are zeros in the normal case, but they are nonzeros in the neighborhood of algorithmic and kinematic singularities. This property can be a good alternative to choose the value of dampening factor. Here, the singularity-robust inverse kinematics is proposed by applying the dampening factor in Eq. (6), in other words, the dampening factor is defined as the scaled 2-norm of the Lagrange multiplier vector as follows:

$$\varepsilon^2 = \alpha |\lambda|$$

(21)

where $\alpha$ is a positive constant and $\lambda$ is calculated by using Eq. (19). This is a natural choice for the dampening factor because the Lagrange multiplier vector is nonzero only in the neighborhood of singularities. In the following section, the effectiveness and implementation method of the proposed singularity-robust inverse kinematics will be shown through the numerical simulation.

3 Simulation

As a numerical simulation, the dual arm manipulator system is adopted to show the effectiveness of the suggested singularity-robust inverse kinematics algorithm for two tasks with priority order. The desired tasks are deburring and conveyance of workpiece, as shown in Fig. 2. The left arm grips a workpiece to be deburred, and the right arm is equipped with a tool for a deburring task. At the same time, the conveyance task of workpiece is implemented as a secondary task. The primary task requires three degrees of freedom for the implementation of a 150 deg deburring task including tool orientation in local $x$-$y$ coordinates, and the secondary task requires two degrees of freedom for a conveyance task from $(-0.2, 0.2)$ to $(0.6, 0.6)$ in global $X$-$Y$ coordinates; in other words, desired tasks can be defined as follows:

1. Primary task: deburring for workpiece ($R_1 = J_1 q$)
2. Secondary task: conveyance of workpiece ($R_2 = J_2 q$)

where $R_1 \in \mathbb{R}^3$, $R_2 \in \mathbb{R}^2$, and $q = [q_{\text{left}}^T q_{\text{right}}^T] \in \mathbb{R}^6$. The kinematic parameters and initial configurations of left and right arms are given by Table 1.

The simulation results are shown in Fig. 3 with various $\alpha$ values according to time progress, where arm configurations were captured from data at times $0 \text{ s}, 0.25 \text{ s}, 0.5 \text{ s}, 0.75 \text{ s},$ and $1 \text{ s}$. The primary and secondary task execution performances are shown in Figs. 4 and 5, respectively. Here, we should note that the primary task is expressed in local $x$-$y$ coordinates fixed in the center of the workpiece, and the secondary task in global $X$-$Y$ coordinates. Though the dampening factors are changed with $\alpha = 0.0001$, $\alpha = 0.1$, and $\alpha = 100$, the primary task is performed well within about $\pm 0.0015 \text{ m}$ and $\pm 0.0003 \text{ rad}$ errors, as shown in Fig. 4.
small errors were caused by the numerical integration. However, the secondary task errors are inevitable because the desired secondary task itself is outside of the workspace of the manipulator. Moreover, the secondary task performance is affected according to the dampening factor $\alpha$, as shown in Fig. 5. The smaller $\alpha$ is, the better the secondary task performance in the viewpoint of an error 2-norm is. Also, nonzero Lagrange multipliers started to exist from the configurations where secondary task could not be implemented, as shown in Fig. 6. These Lagrange multiplier vectors were used for the dampening factor. In the suggested algorithms, we should note that the dampening factor is not a constant.

<table>
<thead>
<tr>
<th>No. of joints</th>
<th>Link length (m)</th>
<th>Initial configuration (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left arm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>First</td>
<td>0.30</td>
<td>$-180.00$</td>
</tr>
<tr>
<td>Second</td>
<td>0.25</td>
<td>$-90.96$</td>
</tr>
<tr>
<td>Third</td>
<td>0.20</td>
<td>$-98.63$</td>
</tr>
<tr>
<td>Right arm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>First</td>
<td>0.30</td>
<td>$110.00$</td>
</tr>
<tr>
<td>Second</td>
<td>0.25</td>
<td>21.42</td>
</tr>
<tr>
<td>Third</td>
<td>0.20</td>
<td>112.48</td>
</tr>
</tbody>
</table>

Fig. 2 Desired tasks for dual arm manipulation, where (a) primary task (de-burring) and (b) secondary task (conveyance)

Fig. 3 Configurations of dual arms according to time progress, where (a) when $\alpha=0.0001$, (b) when $\alpha=0.1$, and (c) when $\alpha=100$

Fig. 4 Execution performances of primary task, where $o$ error means the tool orientation error, (a) when $\alpha=0.0001$, (b) when $\alpha=0.1$, and (c) when $\alpha=100$
but a variable, which is determined by Eq. (21). As shown in the simulation results, the suggested singularity-robust inverse kinematics algorithm is robust and keeps the task-priority well in the neighborhood of singularities.

For a comparative study, the well-known algorithm of Maciejewski’s and Klein [5] or Nakamura et al. [6] is used to show the effectiveness of the suggested algorithm. Both algorithms in Refs. [5,6] are the same, although the algorithm of Nakamura et al. was published later than that of Maciejewski and Klein. It was known that the latter’s algorithm had been published under the review of the algorithm of Nakamura et al. As a matter of fact, the original papers of Maciejewski and Klein and Nakamura et al. did not deal with the singular cases, so the comparison of algorithms may be meaningless because the suggested algorithms are quite different from the conventional method. Also, since Wampler [9] introduced the concept of damping factor, many dampening methods...
have been suggested until now. Thus, the task execution performances become different according to the utilized dampening methods. At the same condition with the suggested algorithm, the algorithm of Maciejewski and Klein or that of Nakamura et al. showed a bad performance, as shown in Fig. 7. However, we could find similar simulation results with the suggested algorithm by enlarging the SVD tolerance for matrix inversion, as shown in Fig. 8.

4 Concluding Remark

In this paper, the inverse kinematics problem was formulated to be an optimization one subject to an equality constraint, in other words, to be a minimization problem of secondary task error subject to an equality constraint for primary task execution. Second, in the procedure of minimization for a given objective function, a new inverse kinematics algorithm was derived including the Lagrange multiplier. Third, in the neighborhood of singular configuration, since nonzero Lagrange multiplier values appear, we chose them as a natural choice of the dampening factor to alleviate the ill-conditioning of matrix inversion, ultimately for singularity-robust inverse kinematics. Finally, the effectiveness and robustness of the suggested singularity-robust inverse kinematics were shown through the numerical simulation about deburring and conveyance tasks of the dual arm system.

Acknowledgment

This work was supported in part by the Bridge Inspection Robot Development Interface (BIRDI) Program under Grant KICT-TEP and MOCT and in part by the Information Technology (IT) Research and Development Program under Grant IITA and MIC, Republic of Korea.

References