Solutions of Final Exam

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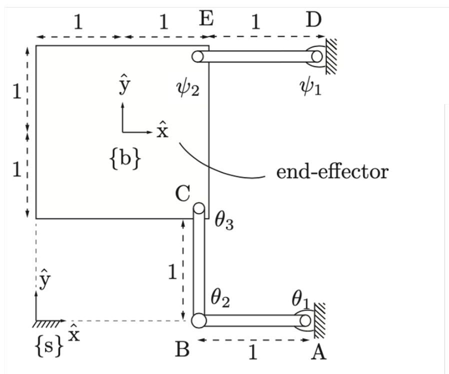
Notice that the answers should be written only in English, otherwise you will get a zero point.

Problem 1 (30pt) Assume a five-bar linkage is in its zero position. Let (p_x, p_y) be the position of the {b}-frame origin expressed in {s}-frame coordinates, and let ϕ be the orientation of the {b} frame. Find the forward kinematics Jacobian J_a from $\mathcal{V}_s = J_a \dot{q}_a$ when B, D are actuated, i.e, $\dot{q}_a = (\dot{\theta}_2, \dot{\psi}_1)$ and $\dot{q}_p = (\dot{\theta}_1, \dot{\theta}_3, \dot{\psi}_2)$. You may make use of the matrix inversion of $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ -3 & -2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & -1 \\ 4.5 & -0.5 & 1.5 \\ 1.5 & -0.5 & 0.5 \end{bmatrix}$.



Solution of Problem 1 (30pt) Since

$$\begin{aligned} \mathcal{V}_s &= J_1 \dot{\theta} = J_2 \dot{\psi} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ -3 & -2 & -2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 3 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{bmatrix} \end{aligned}$$

we can obtain the following relations

$$J_1 \dot{\theta} - J_2 \dot{\psi} = 0 \qquad\qquad H_a \dot{q}_a + H_p \dot{q}_p = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & -3 & -3 \\ -3 & -2 & -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\psi}_1 \\ \dot{\psi}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ 0 & -3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_2 \\ \dot{\psi}_1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ -3 & -2 & 2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_3 \\ \dot{\psi}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, we have

$$\dot{q}_p = -H_p^{-1} H_a \dot{q}_a$$

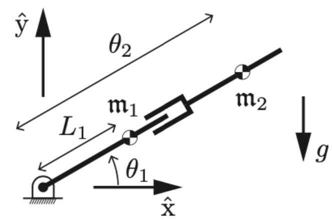
$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_3 \\ \dot{\psi}_2 \end{bmatrix} = -\begin{bmatrix} -2 & 0 & -1 \\ 4.5 & -0.5 & 1.5 \\ 1.5 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_2 \\ \dot{\psi}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.5 & -1.5 \\ -0.5 & -1.5 \end{bmatrix} \begin{bmatrix} \dot{\theta}_2 \\ \dot{\psi}_1 \end{bmatrix}$$

Thus

$$\begin{aligned}
\mathcal{V}_s &= J_1 \dot{\theta} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ -3 & -2 & -2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ -3 & -2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1.5 & -1.5 \end{bmatrix} \begin{bmatrix} \dot{\theta}_2 \\ \dot{\psi}_1 \end{bmatrix} \\
&= \begin{bmatrix} -0.5 & -0.5 \\ -1.5 & -1.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_2 \\ \dot{\psi}_1 \end{bmatrix} \\
&= J_a \dot{q}_a
\end{aligned}$$

Problem 2 (40pt) The figure illustrates an RP robot moving in a vertical plane. The mass of link 1 is $\mathrm{m}_1 = 1[kg]$ and the center of mass is a distance $L_1 = 1[m]$ from joint 1. The scalar inertia of link 1 about an axis through the center of mass and out of the plane is $\mathcal{I}_1 = 1[kg \cdot m^2]$. The mass of link 2 is $\mathrm{m}_2 = 1[kg]$, the center of mass is a distance θ_2 from joint 1, and the scalar inertia of link 2 about its center of mass is $\mathcal{I}_2 = 1[kg \cdot m^2]$. Gravity g acts downward on the page. Derive the equation of motion ?

$$M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta) = \tau$$



Solution of Problem 2 (40pt) Positions of the center of mass of link i

$$x_1 = L_1 \cos\theta_1 = \cos\theta_1 \qquad y_1 = L_1 \sin\theta_1 = \sin\theta_1 \qquad x_2 = \theta_2 \cos\theta_1 \qquad y_2 = \theta_2 \sin\theta_1$$

Velocities of the center of mass of link i

$$\dot{x}_1 = -\dot{\theta}_1 \sin\theta_1 \qquad \dot{y}_1 = \dot{\theta}_1 \cos\theta_1 \qquad \dot{x}_2 = \dot{\theta}_2 \cos\theta_1 - \theta_2\dot{\theta}_1 \sin\theta_1 \qquad \dot{y}_2 = \dot{\theta}_2 \sin\theta_1 + \theta_2\dot{\theta}_1 \cos\theta_1$$

Kinetic energies of link i

$$\mathcal{K}_1 = \frac{1}{2}\mathrm{m}_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}\mathcal{I}_1\dot{\theta}_1^2 = \dot{\theta}_1^2$$
$$\mathcal{K}_2 = \frac{1}{2}\mathrm{m}_2(\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}\mathcal{I}_2\dot{\theta}_1^2 = \frac{1}{2}\left((1 + \theta_2^2)\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$$

Potential energies of link i

$$\mathcal{P}_1 = \mathrm{m}_1 g y_1 = g \sin\theta_1$$
$$\mathcal{P}_2 = \mathrm{m}_2 g y_2 = g\theta_2 \sin\theta_1$$

Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= \mathcal{K}_1 + \mathcal{K}_2 - \mathcal{P}_1 - \mathcal{P}_2 \\ &= \frac{3}{2}\dot{\theta}_1^2 + \frac{1}{2}\theta_2^2\dot{\theta}_1^2 + \frac{1}{2}\dot{\theta}_2^2 - g\sin\theta_1 - g\theta_2\sin\theta_1\end{aligned}$$

The Euler-Lagrange equations for this example are of the form

$$\begin{aligned}
\tau_1 &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} - \frac{\partial \mathcal{L}}{\partial \theta_1} \\
&= \frac{d}{dt}(3\dot{\theta}_1 + \theta_2^2\dot{\theta}_1) + g\cos\theta_1 + g\theta_2\cos\theta_1 \\
&= 3\ddot{\theta}_1 + 2\theta_2\dot{\theta}_2\dot{\theta}_1 + \theta_2^2\ddot{\theta}_1 + g\cos\theta_1 + g\theta_2\cos\theta_1 \\
\tau_2 &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} - \frac{\partial \mathcal{L}}{\partial \theta_2} \\
&= \frac{d}{dt}(\dot{\theta}_2) - \theta_2\dot{\theta}_1^2 + g\sin\theta_1 \\
&= \ddot{\theta}_2 - \theta_2\dot{\theta}_1^2 + g\sin\theta_1
\end{aligned}$$

Now we can complete the dynamics as follows:

$$M(\theta)\ddot{\theta} + c(\theta,\dot{\theta}) + g(\theta) = \tau$$

$$\begin{bmatrix} 3+\theta_2^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2\theta_2\dot{\theta}_1\dot{\theta}_2 \\ -\theta_2\dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} (1+\theta_2)g\cos\theta_1 \\ g\sin\theta_1 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Problem 3 (30pt) For given a one-dof mass-spring-damper system of the form $\mathrm{m}\ddot{x} + b\dot{x} + kx = f$, where f is the control force and $\mathrm{m} = 4[kg]$ and $b = 2[Ns/m]$ and $k = 0.1[N/m]$, determine the gains k_p and k_d so that the following PD controller

$$f = k_d(\dot{x}_d - \dot{x}) + k_p(x_d - x)$$

can yield the critical damping and the 2% settling time of $0.01[s]$, where $x_d = 1$ and $\dot{x}_d = 0$.

Solution of Problem 3 (30pt) Let us obtain the error dynamics using $x_e = x_d - x$, $\dot{x}_e = -\dot{x}$, and $\ddot{x}_e = -\ddot{x}$

$$\begin{aligned} \mathrm{m}\ddot{x} + b\dot{x} + kx &= k_d(\dot{x}_d - \dot{x}) + k_p(x_d - x) \\ -\mathrm{m}\ddot{x}_e - b\dot{x}_e + k(x_d - x_e) &= k_d\dot{x}_e + k_p x_e \\ 4\ddot{x}_e + (k_d + 2)\dot{x}_e + (k_p + 0.1)x_e &= 0.1 \\ \ddot{x}_e + \frac{k_d + 2}{4}\dot{x}_e + \frac{k_p + 0.1}{4}x_e &= 0.025 \end{aligned}$$

From the following relations, we have

$$\omega_n^2 = \frac{k_p + 0.1}{4} \qquad\qquad 2\zeta\omega_n = \frac{k_d + 2}{4}$$

Since $\zeta = 1$ and $\frac{4}{\zeta\omega_n} = 0.01$, we have

$$k_p = 639,999.9 \approx 6.4 \times 10^5 \qquad\qquad k_d = 3,198$$