3.2 Twists

Consider both the linear and angular velocities of a moving frame. Let

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0_{3\times 1} & 1 \end{bmatrix}$$

denote the configuration of {b} as seen from {s}.

(In the previous lecture)
Let $R(t) = R_{sb}$ denote the orientation of the rotating frame as seen from the fixed frame. Denote by w the angular velocity of the rotating frame. Then

$$\dot{R}R^{-1} = \dot{R}_{sb}R_{sb}^{-1} = \dot{R}_{s\not{b}}R_{\not{b}s} = [\omega_s] \qquad R^{-1}\dot{R} = R_{sb}^{-1}\dot{R}_{sb} = R_{b\not{s}}\dot{R}_{\not{s}b} = [\omega_b]$$

- $\omega_s \in \Re^3$ is the fixed-frame vector representation of w and $[\omega_s] \in so(3)$ is its 3×3 matrix representation.
- $\omega_b \in \Re^3$ is the body-frame vector representation of w

- Let us first see what happens when we pre-multiply $\dot{T}$ by T^{-1}:

$$T^{-1}\dot{T} = \begin{bmatrix} R^T & -R^T p \\ 0_{3\times 1} & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0_{3\times 1} & 0 \end{bmatrix} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0_{3\times 1} & 0 \end{bmatrix} = \begin{bmatrix} [\omega_b] & v_b \\ 0_{3\times 1} & 0 \end{bmatrix} \quad \leftarrow \quad T_{sb}^{-1}\dot{T}_{sb} = T_{b\not{s}}\dot{T}_{\not{s}b} \ \textbf{body-frame}$$

$$R^T\dot{R} = R^{-1}\dot{R} = R_{sb}^{-1}\dot{R}_{sb} = R_{b\not{s}}\dot{R}_{\not{s}b} = [\omega_b] \qquad\qquad R^T\dot{p} = R^{-1}\dot{p} = R_{sb}^{-1}\dot{p}_s = R_{b\not{s}}\dot{p}_{\not{s}} = \dot{p}_b = v_b$$

- $T^{-1}\dot{T}$ represents the linear and angular velocities of the moving frame relative to the stationary frame {b} currently aligned with the moving frame.
- It is reasonable to merge ω_b and v_b into a single six-dimensional velocity vector.
- Spatial velocity in the body frame, or simply the body-twist, to be

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \Re^6$$

- It is convenient to have a matrix representation of a twist:

$$T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0_{3\times 1} & 0 \end{bmatrix} \in se(3)$$

  where $[\omega_b] \in so(3)$ and $v_b \in \Re^3$. $se(3)$ is called the Lie algebra of the Lie group $SE(3)$.
- $[\mathcal{V}_b] \in se(3)$ represents the matrix representation of the twist $\mathcal{V}_b$ associated with the rigid-body configuration $T \in SE(3)$.

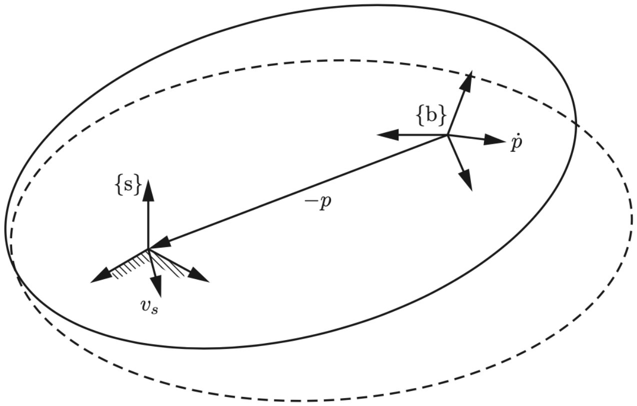

Figure 3.17: Physical interpretation of v_s. The initial (solid line) and displaced (dashed line) configurations of a rigid body.

- Now that we have a physical interpretation for $T^{-1}\dot{T}$, let us evaluate $\dot{T}T^{-1}$:

$$\dot{T}T^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0_{3\times 1} & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0_{3\times 1} & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}R^T & \dot{p} - \dot{R}R^T p \\ 0_{3\times 1} & 0 \end{bmatrix} = \begin{bmatrix} [\omega_s] & v_s \\ 0_{3\times 1} & 0 \end{bmatrix} \quad \leftarrow \quad \dot{T}_{sb}T_{sb}^{-1} = \dot{T}_{s\not{b}}T_{\not{b}s} \quad \textbf{fixed-frame}$$

$$\dot{R}R^T = \dot{R}R^{-1} = \dot{R}_{sb}R_{sb}^{-1} = \dot{R}_{s\not{b}}R_{\not{b}s} = [\omega_s] \qquad \dot{p} - \dot{R}R^T p = \dot{p} - [\omega_s]p = \dot{p} - \omega_s \times p = \dot{p} + \omega_s \times (-p) = v_s$$

- The physical meaning of v_s can now be inferred: imagining the moving body to be infinitely large, v_s is the instantaneous velocity of the point on this body currently at the fixed-frame origin, expressed in the fixed frame (see Figure 3.17).

- Spatial velocity in the space frame, or simply the spatial-twist, is

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \Re^6 \qquad\qquad \dot{T}T^{-1} = [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0_{3\times 1} & 0 \end{bmatrix} \in se(3)$$

- If we regard the moving body as being infinitely large, there is an appealing and natural symmetry between $\mathcal{V}_s = (\omega_s, v_s)$ and $\mathcal{V}_b = (\omega_b, v_b)$:
  - ω_b is the angular velocity expressed in {b}
  - ω_s is the angular velocity expressed in {s}
  - v_b is the linear velocity of a point at the origin of {b} expressed in {b}
  - v_s is the linear velocity of a point at the origin of {s} expressed in {s}
- The relationship b/w $\mathcal{V}_b$ from $\mathcal{V}_s$

$$
\begin{aligned}
[\mathcal{V}_b] &= T^{-1}\dot{T} \qquad \leftarrow \qquad \dot{T}T^{-1} = [\mathcal{V}_s] \\
&= T^{-1}[\mathcal{V}_s]T = T_{b\not{s}}[\mathcal{V}_{\not{s}}]T_{\not{s}b}
\end{aligned}
\qquad\qquad
\begin{aligned}
[\mathcal{V}_s] &= \dot{T}T^{-1} \qquad \leftarrow \qquad T^{-1}\dot{T} = [\mathcal{V}_b] \\
&= T[\mathcal{V}_b]T^{-1} = T_{s\not{b}}[\mathcal{V}_{\not{b}}]T_{\not{b}s}
\end{aligned}
$$

- Consider $[\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1}$

$$
[\mathcal{V}_s] = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R[\omega_b]R^T & -R[\omega_b]R^T p + Rv_b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [R\omega_b] & [p](R\omega_b) + Rv_b \\ 0 & 0 \end{bmatrix}
$$

using $R[\omega]R^T = [R\omega]$ and $[\omega]p = -[p]\omega$, we have the relationship b/w $\mathcal{V}_s$ and $\mathcal{V}_b$:

$$
\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0_{3\times 3} \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}
$$

where the 6×6 matrix pre-multiplying $\mathcal{V}_b$ is useful for changing the frame of reference for twists and wrenches.

Definition 3.6. *Given* $T = (R, p) \in SE(3)$*, its adjoint representation* $[Ad_T]$ *is*

$$[Ad_T] = \begin{bmatrix} R & 0_{3\times 3} \\ [p]R & R \end{bmatrix} \in \Re^{6\times 6}$$

For any $\mathcal{V} \in \Re^6$, the adjoint map associated with T is

$$\begin{aligned} \mathcal{V}' &= [Ad_T]\mathcal{V} \\ &= Ad_T(\mathcal{V}) \end{aligned} \qquad \begin{aligned} \mathcal{V}_s &= [Ad_{T_{sb}}]\mathcal{V}_b \\ &= Ad_{T_{s\cancel{b}}}(\mathcal{V}_{\cancel{b}}) \end{aligned} \qquad \begin{aligned} \mathcal{V}_b &= [Ad_{T_{bs}}]\mathcal{V}_s \\ &= Ad_{T_{b\cancel{s}}}(\mathcal{V}_{\cancel{s}}) \end{aligned}$$

In terms of the matrix form $[\mathcal{V}] \in se(3)$ of $\mathcal{V} \in \Re^6$,

$$[\mathcal{V}'] = T[\mathcal{V}]T^{-1} \qquad \begin{aligned} [\mathcal{V}_s] &= T_{sb}[\mathcal{V}_b]T_{bs} \\ &= T[\mathcal{V}_b]T^{-1} \end{aligned} \qquad \begin{aligned} []\mathcal{V}_b] &= T_{bs}[\mathcal{V}_s]T_{sb} \\ &= T^{-1}[\mathcal{V}_s]T \end{aligned}$$

Proposition 3.12. *Let* $T_1, T_2 \in SE(3)$ *and* $\mathcal{V} = (\omega, v)$. *Then*

$$[Ad_{T_1}][Ad_{T_2}]\mathcal{V} = [Ad_{T_1 T_2}]\mathcal{V}$$
$$Ad_{T_1}(Ad_{T_2}(\mathcal{V})) = Ad_{T_1 T_2}(\mathcal{V})$$

Also, for any $T \in SE(3)$ *the following holds:*

$$[Ad_T]^{-1} = [Ad_{T^{-1}}]$$
$$\begin{bmatrix} R & 0_{3\times 3} \\ [p]R & R \end{bmatrix}^{-1} = \begin{bmatrix} R^T & 0_{3\times 3} \\ -R^T[p] & R^T \end{bmatrix}$$

when $T = (R, p)$.

The second property follows from the first on choosing $T_1 = T^{-1}$ and $T_2 = T$, so that

$$Ad_{T^{-1}}(Ad_T(\mathcal{V})) = Ad_{T^{-1}T}(\mathcal{V}) = Ad_I(\mathcal{V}) = \mathcal{V}$$

Again analogously to the case of angular velocities, it is important to realize that, for a given twist, its fixed-frame representation $\mathcal{V}_s$ does not depend on the choice of the body frame {b}, and its body-frame representation $\mathcal{V}_b$ does not depend on the choice of the fixed frame {s}.

For Summary of Results on Twists, please read Proposition 3.22 in the textbook.

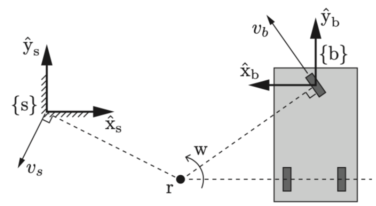

Figure 3.18: The twist corresponding to the instantaneous motion of the chassis of a three-wheeled vehicle can be visualized as an angular velocity w about the point r.

Example 3.3. *Consider a top view of a car, with a single steerable front wheel, driving on a plane. The angle of the front wheel of the car causes the car's motion to be a pure angular velocity* $\mathrm{w} = 2rad/s$ *about an axis out of the page at the point* r *in the plane. Inspecting the figure, we can write* r *as* $r_s = (2, -1, 0)$ *or* $r_b = (2, -1.4, 0)$*,* w *as* $\omega_s = (0, 0, 2)$ *or* $\omega_b = (0, 0, -2)$*, and* T_{sb} *as*

$$T_{sb} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0.4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the figure and simple geometry, we get $v_s = \omega_s \times (-r_s) = r_s \times \omega_s = [r_s]\omega_s = (-2, -4, 0)$, $v_b = \omega_b \times (-r_b) = r_b \times \omega_b = [r_b]\omega_b = (2.8, 4, 0)$, and thus obtain the twists $\mathcal{V}_s$ and $\mathcal{V}_b$:

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ -4 \\ 0 \end{bmatrix} \qquad \mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2.8 \\ 4 \\ 0 \end{bmatrix} \qquad \mathcal{V}_s = [Ad_{T_{sb}}]\mathcal{V}_b = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & -1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 \\ 0.4 & 4 & 0 & 0 & 0 & -1 \end{bmatrix} \mathcal{V}_b$$

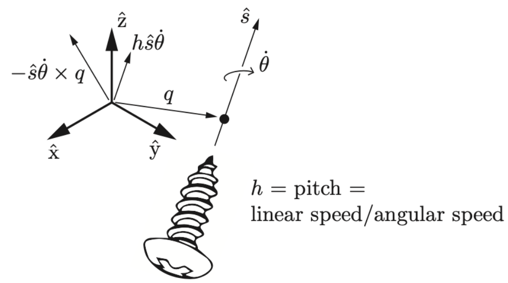

Figure 3.19: A screw axis $\mathcal{S}$ represented by a point q, a unit direction $\hat{s}$, and a pitch h.

The Screw Interpretation of a Twist

- An angular velocity ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis.
- A twist $\mathcal{V}$ can be interpreted in terms of a screw axis $\mathcal{S}$ and a velocity $\dot{\theta}$ about the screw axis.
- A screw axis represents the familiar motion of a screw: rotating about the axis while also translating along the axis. One representation of a screw axis $\mathcal{S}$ is the collection $\{q, \hat{s}, h\}$, where
  - $q \in \Re^3$ is any point on the axis,
  - $\hat{s}$ is a unit vector in the direction of the axis, and
  - h is the screw pitch, which defines the ratio of the linear velocity along the screw axis to the angular velocity $\dot{\theta}$ about the screw axis.
- Using the geometry, we can write the twist $\mathcal{V} = (\omega, v)$ corresponding to an angular velocity $\dot{\theta}$ about $\mathcal{S}$ (represented by $\{q, \hat{s}, h\}$) as

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \omega \\ \omega \times (-q) + h\omega \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ \hat{s}\dot{\theta} \times (-q) + h\hat{s}\dot{\theta} \end{bmatrix} = \begin{bmatrix} \hat{s} \\ \hat{s} \times (-q) + h\hat{s} \end{bmatrix} \dot{\theta} = \mathcal{S}\dot{\theta}$$

- Instead of representing the screw axis $\mathcal{S}$ using the cumbersome collection $\{q, \hat{s}, h\}$, let us define the screw axis $\mathcal{S}$ using a normalized version of any twist $\mathcal{V} = (\omega, v)$ corresponding to motion along the screw:
  - If $\omega \neq 0$ then the screw axis $\mathcal{S}$ is simply $\mathcal{V}$ normalized by the length of the angular velocity vector ω. The angular velocity about the screw axis is $\dot{\theta} = \|\omega\|$, such that $\mathcal{S}\dot{\theta} = \mathcal{V}$

$$\mathcal{S} = \frac{\mathcal{V}}{\|\omega\|} = \left(\frac{\omega}{\|\omega\|}, \frac{v}{\|\omega\|}\right)$$

  - If $\omega = 0$ then the screw axis $\mathcal{S}$ is simply $\mathcal{V}$ normalized by the length of the linear velocity vector. The linear velocity along the screw axis is $\dot{\theta} = \|v\|$, such that $\mathcal{S}\dot{\theta} = \mathcal{V}$

$$\mathcal{S} = \frac{\mathcal{V}}{\|v\|} = \left(0, \frac{v}{\|v\|}\right)$$

- This leads to the following definition of a unit (normalized) screw axis (Read Definition 3.24 in the textbook)
- Since a screw axis represented as $\mathcal{S}_a$ in a frame {a} is related to the representation $\mathcal{S}_b$ in a frame {b} by

$$\mathcal{S}_a = [Ad_{T_{ab}}]\mathcal{S}_b \qquad \mathcal{S}_b = [Ad_{T_{ba}}]\mathcal{S}_a$$