4 Heat and Fluid-Flow Models

1. Heat Flow

   - The dynamic models of temperature control systems involve the flow and storage of heat energy.
   - Heat energy flows (q) through substances at a rate proportional to the temperature difference across the substance; that is,

   $$q = \frac{1}{R}(T_1 - T_2) \quad \leftrightarrow \quad i = \frac{V_1 - V_2}{R}$$

   - The net heat energy flows into a substance affects the temperature of the substance according to the relation:

   $$q = C\dot{T} \quad \leftrightarrow \quad i = C\dot{V}$$

   - The heat can also flow when a warmer mass flows into a cooler mass, or vice versa.

   $$q = wc_v(T_1 - T_2)$$

   where

   q: heat-energy flow

   T: temperature

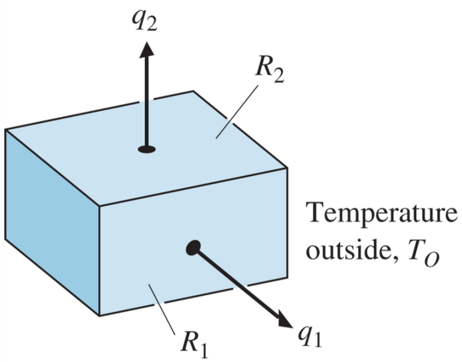
   w: mass flow rate $= \dot{m}$

   k: thermal conductivity

   R: thermal resistance $= \dfrac{l}{kA}$

   C: thermal capacity $= mc_v$

   c_v: specific heat



(Example 2.16, Heat Flow from a Room) For a room with all but two sides insulated ($R = \infty$), Find the differential equations that determine the temperature in the room

$$-q_1 - q_2 = C_I \dot{T}_I$$
$$-\frac{T_I - T_o}{R_1} - \frac{T_I - T_o}{R_2} = C_I \dot{T}_I$$

where

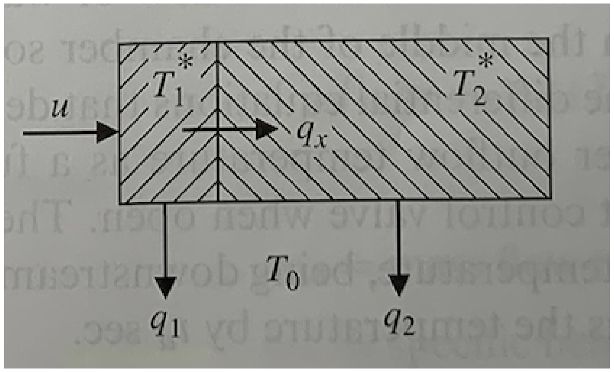
T_I: temperature inside

T_o: temperature outside

R_1: thermal resistance of the room wall

R_2: thermal resistance of the room ceiling

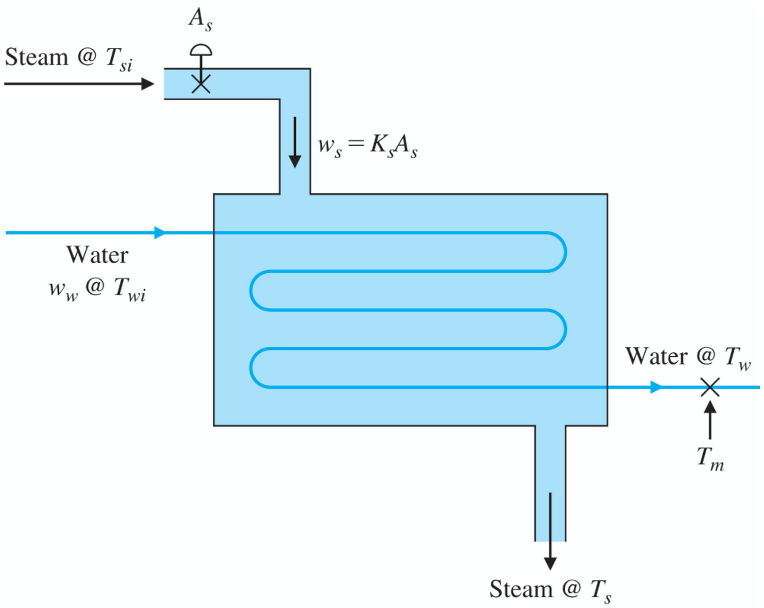
C_I: thermal capacity of air within the room



(Example 2.17, A Thermal Control System) Thermal masses are contacted each other, where heat is being applied to the mass on the left. There is also heat transferred directly to the second mass in contact with it, and heat is lost to the environment from both masses. Find the relevant differential equations?

$$u - q_1 - q_x = C_1\dot{T}_1 \qquad q_x - q_2 = C_2\dot{T}_2$$

$$u - \frac{T_1 - T_0}{R_1} - \frac{T_1 - T_2}{R_x} = C_1\dot{T}_1 \qquad \frac{T_1 - T_2}{R_x} - \frac{T_2 - T_0}{R_2} = C_2\dot{T}_2$$



(Example 2.18, Modeling a Heat Exchanger) Steam enters the chamber through the controllable valve at the top, and cooler steam leaves at the bottom. Find the differential equations ?

- Heat into chamber from the inlet steam and Heat into the outlet water from chamber

$$q_{in} = w_s c_{vs}(T_{si} - T_s) \qquad q_{out} = w_w c_{vw}(T_w - T_{wi})$$

- The rates of temperatures of the steam and the water

$$q_{in} - q_{sw} = C_s \dot{T}_s \qquad q_{sw} + q_{out} = C_w \dot{T}_w$$

$$w_s c_{vs}(T_{si} - T_s) - \frac{T_s - T_w}{R} = C_s \dot{T}_s \qquad \frac{T_s - T_w}{R} + w_w c_{vw}(T_{wi} - T_w) = C_w \dot{T}_w$$

2. Incompressible Fluid Flow

- Hydraulic actuator is used extensively in control system bc it can supply a large force with low inertia and low weight.
- The physical relations governing fluid flow are continuity, force equilibrium, and resistance.
- The continuity:

$$\dot{m} = w_{in} - w_{out} \qquad \text{fluid mass = mass inflow rate – mass outflow rate}$$

- Force equilibrium:

$$f = pA \qquad \text{force = pressure in the fluid} \times \text{area on which the fluid acts}$$

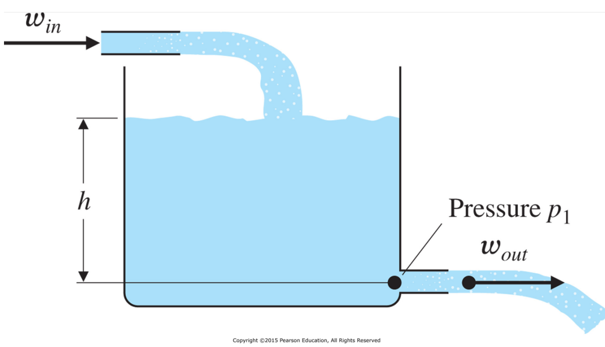
- Mass flow rate by resistance: (the flow is resisted either by a constriction in the path or by friction)

$$w = \frac{1}{R}(p_1 - p_2)^{1/\alpha}$$

  where w is mass flow rate and α constant whose value depends on the type of restriction. The constant α takes on values between 1 and 2. For example, $\alpha = 2$ for high flow rates and $\alpha = 1$ for very slow flow rates.
- Volume flow rate:

$$Q = \frac{1}{\rho R}(p_1 - p_2)^{1/\alpha}$$

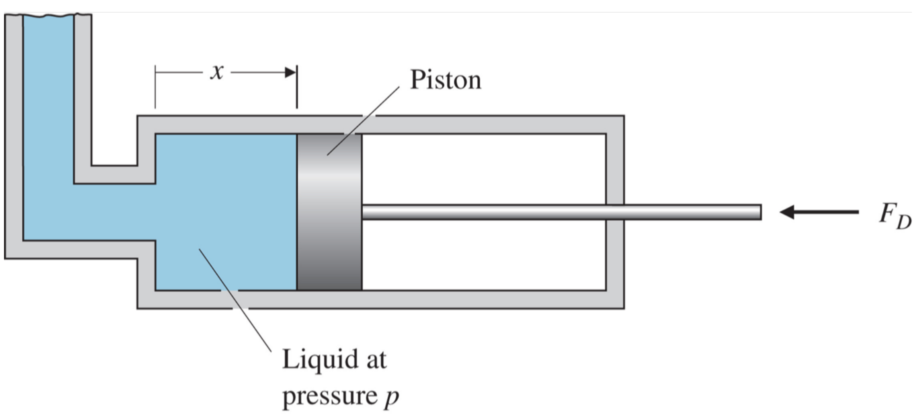
  where Q is volume flow rate $Q = w/\rho$ and ρ is fluid density.



(Example 2-19, Water Tank Height) A area of the tank, ρ density of water, h height of water, and m mass of water in the tank

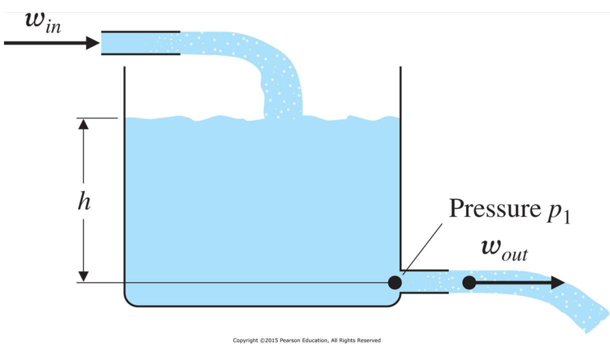
$$\dot{m} = \rho\dot{V} = \rho(A\dot{h}) = w_{in} - w_{out} \qquad \rightarrow \qquad \dot{h} = \frac{1}{\rho A}(w_{in} - w_{out})$$

It is noted that the density $\rho = \frac{m}{V}$ implies mass per volume.



(Example 2-20, Modeling of Hydraulic Piston) Assume mass of the piston is m.

$$m\ddot{x} = Ap - F_D$$



(Example 2.21, Water Tank Height and Outflow) Assume there is relatively short restriction at the outlet and that $\alpha = 2$. Find the nonlinear differential equation describing the height of the water in the tank?

- Flow out of the tank:

$$w_{out} = \frac{1}{R}(p_1 - p_a)^{\frac{1}{2}}$$

  where $p_1 = \rho g h + p_a$ and p_a is ambient pressure

- Height:

$$\begin{aligned} \dot{h} &= \frac{1}{\rho A}(w_{in} - w_{out}) \\ &= \frac{1}{\rho A}\left(w_{in} - \frac{\sqrt{p_1 - p_a}}{R}\right) \end{aligned}$$

1. See Table 2.1 for key equations for dynamic models.]

Key Equations for Dynamic Models

System	Important Laws or Relationships	Associated Equations	Equation Number(s)
Mechanical	Translational motion (Newton's law)	$F = m\mathbf{a}$	(2.1)
	Rotational motion	$M = I\alpha$	(2.14)
Electrical	Operational amplifier		(2.46), (2.47)
Electromechanical	Law of motors	$F = Bli$	(2.53)
	Law of generators	$e = Blv$	(2.56)
	Torque developed in a rotor	$T = K_t i_a$	(2.60)
Back emf	Voltage generated as a result of rotation of a rotor	$e = K_e \dot{\theta}_m$	(2.61)
Gears	Effective inertia	$J_{eq} = J_2 + J_1 n^2$	(2.80)
Heat flow	Heat-energy flow	$q = {}^1/_R(T_1 - T_2)$	(2.81)
	Temperature as a function of heat-energy flow	$\dot{T} = \frac{1}{C}q$	(2.82)
	Specific heat	$C = mc_v$	(2.83)
Fluid flow	Continuity relation (conservation of matter)	$\dot{m} = w_{in} - w_{out}$	(2.88)
	Force of a fluid acting on a piston	$f = pA$	(2.90)
	Effect of resistance to fluid flow	$w = {}^1/_R(p_1 - p_2)^{1/\alpha}$	(2.91)

2. (Homework #2, Due : April 5th, 2020) Solve and Submit 2.1, 2.3, 2.8, 2.11, 2.13, 2.15, 2.17, 2.18, 2.21, 2.22, 2.26, 2.27