Solutions of Final Exam

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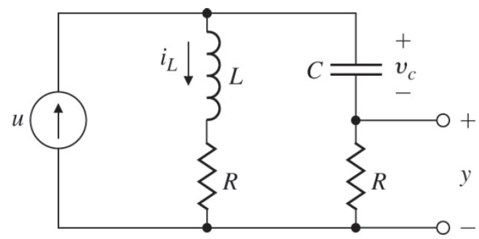
Problem 1 (20pt) Consider the electric circuit shown in the figure.

(1.1) Write the state equations for the circuit, where the input $u(t)$ is a current, and the output $y(t)$ is a voltage. Let $x_1(t) = i_L(t)$ and $x_2(t) = v_c(t)$.

(1.2) What condition(s) on R, L, and C will guarantee that the system is controllable



Solution of Problem 1 (20pt)

(1.1) By applying KCL and KVL, we have

$$u = i_L + C\frac{dv_c}{dt} \qquad L\frac{di_L}{dt} + Ri_L = v_c + RC\frac{dv_c}{dt} \qquad y = RC\frac{dv_c}{dt}$$

$$u = x_1 + C\dot{x}_2 \qquad L\dot{x}_1 + Rx_1 = x_2 + RC\dot{x}_2 \qquad y = RC\dot{x}_2$$

$$\dot{x}_2 = -\frac{1}{C}x_1 + \frac{1}{C}u \qquad \dot{x}_1 = -\frac{2R}{L}x_1 + \frac{1}{L}x_2 + \frac{R}{L}u \qquad y = -Rx_1 + Ru$$

Therefore,

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{2R}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{R}{L} \\ \frac{1}{C} \end{bmatrix} u$$

$$y = \begin{bmatrix} -R & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Ru$$

(1.2) Controllability matrix

$$\mathcal{O} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \frac{R}{L} & -\frac{2R^2}{L^2} + \frac{1}{LC} \\ \frac{1}{C} & -\frac{R}{LC} \end{bmatrix}$$

if the following is satisfied, then the system is controllable

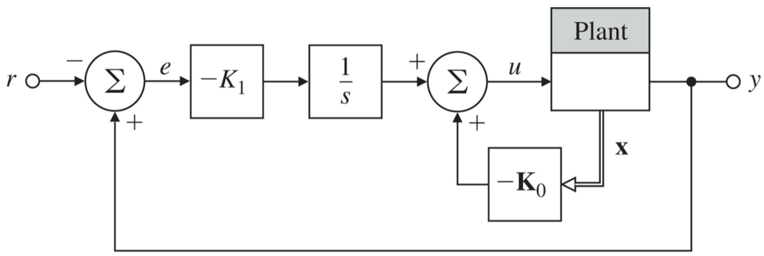
$$\det(\mathcal{O}) = \frac{R^2}{CL^2} - \frac{1}{LC^2} \neq 0 \quad \rightarrow \quad \therefore\ R^2 \neq \frac{L}{C}$$

Problem 2 (25pt) Consider a system with state equation

$$\dot{x} = Ax + Bu \qquad\qquad y = Cx$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$



The system steady-state error performance can be made robust by augmenting the system with an integrator and using unity feedback; that is, by setting $\dot{x}_I = y - r$, where x_I is the state of the integrator. To see this, find state feedback $K_0 = [K_{01}, K_{02}]$ and K_1 of the form $u = -K_0 x - K_1 x_I$ so that the poles of the augmented system are at -3 ; $-2 \pm j3$.

Solution of Problem 2 (25pt)

1. Since $\dot{x}_I = Cx - r$, we have

$$\begin{bmatrix} \dot{x}_I \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & C \\ 0_{1\times 2} & A \end{bmatrix} \begin{bmatrix} x_I \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u - \begin{bmatrix} 1 \\ 0_{1\times 2} \end{bmatrix} r$$

$$\begin{bmatrix} \dot{x}_I \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_I \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r$$

2. Here, since $u = -K_{01}x_1 - K_{02}x_2 - K_1 x_I$, we get

$$\begin{bmatrix} \dot{x}_I \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -K_1 & -K_{01} & -3-K_{02} \end{bmatrix} \begin{bmatrix} x_I \\ x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r$$

3. By using the pole placement,

$$\det \begin{bmatrix} s & -1 & 0 \\ 0 & s+2 & -1 \\ K_1 & K_{01} & s+3+K_{02} \end{bmatrix} = (s+3)(s^2+4s+13)$$

$$s^3 + (5+K_{02})s^2 + (6+2K_{02}+K_{01})s + K_1 = s^3 + 7s^2 + 25s + 39$$

4. Therefore

$$\therefore \quad K_1 = 39 \quad \text{and} \quad K_0 = \begin{bmatrix} 15 & 2 \end{bmatrix}$$

Problem 3 (25pt) Consider the following compensator

$$D_c(s) = \frac{5}{s+5}$$

(3.1) Determine the sampling time T from $\omega_s = 25 \times \omega_{bw}$, where ω_s implies sampling rate and ω_{bw} means a bandwidth.

(3.2) Find the approximate model using Tustin's method ?

(3.3) Find the approximate model using ZOH ?

(3.4) Find the approximate model using MPZ ?

(3.5) Find the approximate model using MMPZ (modified MPZ) ?

Solution of Problem 3 (25pt)

(3.1) Since $\omega_{bw} = 5[rad/s]$, the sampling time should be chosen as

$$T = \frac{2\pi}{\omega_{bw}} = \frac{2\pi}{125} \approx 0.05[s]$$

(3.2) Tustin's method

$$\begin{aligned} D_d(z) &= \frac{5}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + 5} = \frac{5T(1+z^{-1})}{2(1-z^{-1}) + 5T(1+z^{-1})} = \frac{5T + 5Tz^{-1}}{(2+5T) - (2-5T)z^{-1}} \\ &= \left(\frac{5T}{2+5T}\right)\frac{1+z^{-1}}{1-\left(\frac{2-5T}{2+5T}\right)z^{-1}} \approx 0.11\frac{1+z^{-1}}{1-0.78z^{-1}} \end{aligned}$$

(3.3) ZOH

$$\begin{aligned} D_d(z) &= (1-z^{-1})\mathcal{Z}\left(\frac{D_c(s)}{s}\right) = (1-z^{-1})\mathcal{Z}\left(\frac{5}{s(s+5)}\right) = (1-z^{-1})\frac{(1-e^{-5T})z^{-1}}{(1-z^{-1})(1-e^{-5T}z^{-1})} \\ &= (1-e^{-5T})\frac{z^{-1}}{1-e^{-5T}z^{-1}} \approx 0.22\frac{z^{-1}}{1-0.78z^{-1}} \end{aligned}$$

(3.4) MPZ

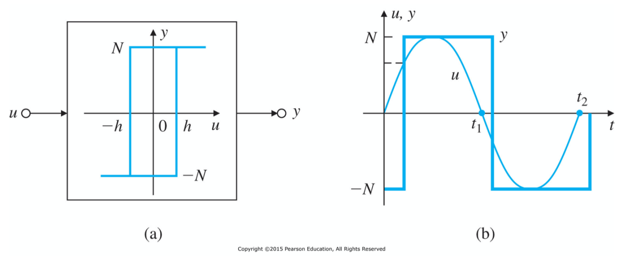
$$\begin{aligned} D_d(z) &= K_d\frac{(1+z^{-1})}{1-e^{-5T}z^{-1}} \quad \text{where } K_d\frac{2}{1-e^{-5T}} = 1 \\ &= \left(\frac{1-e^{-5T}}{2}\right)\frac{1+z^{-1}}{1-e^{-5T}z^{-1}} \approx 0.11\frac{1+z^{-1}}{1-0.78z^{-1}} \end{aligned}$$

(3.5) MMPZ

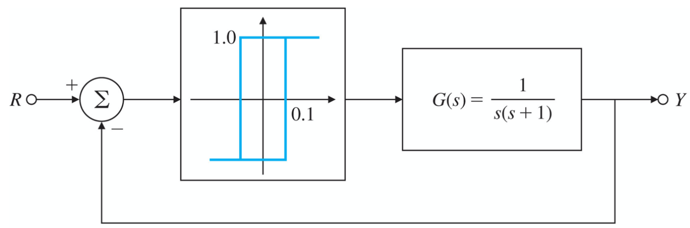
$$\begin{aligned} D_d(z) &= K_d\frac{z^{-1}}{1-e^{-5T}z^{-1}} \quad \text{where } K_d\frac{1}{1-e^{-5T}} = 1 \\ &= (1-e^{-5T})\frac{z^{-1}}{1-e^{-5T}z^{-1}} \approx 0.22\frac{z^{-1}}{1-0.78z^{-1}} \end{aligned}$$

Problem 4 (30pt) Consider the *relay function with hysteresis* shown in the below figure.

(4.1) Find the describing function (equivalent gain) for this nonlinearity when $u = a \sin \omega t$, where the output is a square wave with amplitude N as long as the input amplitude a is greater than the hysteresis level h.



(4.2) Find the amplitude and the frequency of the limit cycle? where $N = 1$ and $h = 0.1$



Solution of Problem 4 (30pt)

(4.1) The describing function is obtained from the first harmonic components as follow:

$$DF = K_{eq}(a) = \frac{b_1 + ja_1}{a}$$

1. From the figure, it is seen that the square wave lags the input in time. The lag time can be calculated as the time when

$$a \sin \omega t = h \qquad \rightarrow \qquad \omega_s t = \sin^{-1} \frac{h}{a}$$

2. Let us calculate a_1 as follow:

$$\begin{aligned}
a_1 &= \frac{2}{\pi} \int_0^{\pi} u(t) \cos(\omega t) d(\omega t) \\
&= \frac{2}{\pi} \int_0^{\omega_s t} u(t) \cos(\omega t) d(\omega t) + \frac{2}{\pi} \int_{\omega_s t}^{\pi} u(t) \cos(\omega t) d(\omega t) \\
&= \frac{2}{\pi} \int_0^{\omega_s t} -N \cos(\omega t) d(\omega t) + \frac{2}{\pi} \int_{\omega_s t}^{\pi} N \cos(\omega t) d(\omega t) \\
&= \frac{2}{\pi} \left[-N \sin \theta|_0^{\omega_s t} + N \sin \theta|_{\omega_s t}^{\pi} \right] \\
&= \frac{2N}{\pi} \left[-\sin \omega_s t + 0 + 0 - \sin \omega_s t \right] = -\frac{4N}{\pi} \frac{h}{a}
\end{aligned}$$

3. Let us calculate b_1 as follow:

$$\begin{aligned} b_1 &= \frac{2}{\pi}\int_0^{\pi} u(t)\sin(\omega t)d(\omega t) \\ &= \frac{2}{\pi}\int_0^{\omega_s t} u(t)\sin(\omega t)d(\omega t) + \frac{2}{\pi}\int_{\omega_s t}^{\pi} u(t)\sin(\omega t)d(\omega t) \\ &= \frac{2}{\pi}\int_0^{\omega_s t} -N\sin(\omega t)d(\omega t) + \frac{2}{\pi}\int_{\omega_s t}^{\pi} N\sin(\omega t)d(\omega t) \\ &= \frac{2}{\pi}\left[N\cos\theta|_0^{\omega_s t} - N\cos\theta|_{\omega_s t}^{\pi}\right] \\ &= \frac{2N}{\pi}\left[\cos\omega_s t - 1 + 1 + \cos\omega_s t\right] = \frac{4N}{\pi}\sqrt{1-\frac{h^2}{a^2}} \end{aligned}$$

4. We finally obtain

$$\therefore \qquad K_{eq}(a) = \frac{4N}{a\pi}\left[\sqrt{1-\frac{h^2}{a^2}} - j\frac{h}{a}\right]$$

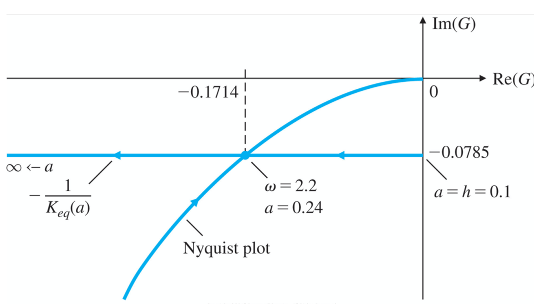
(4.2) The characteristic equation for stability is as follow:

$$1 + K_{eq}(a)G(s) = 0 \quad \rightarrow \quad G(j\omega) = -\frac{1}{K_{eq}(a)}$$

1. The negative reciprocal of the describing function for the hysteresis nonlinearity is

$$-\frac{1}{K_{eq}(a)} = -\frac{1}{\frac{4N}{a\pi}\left[\sqrt{1-\frac{h^2}{a^2}} - j\frac{h}{a}\right]} = -\frac{\pi}{4N}[\sqrt{a^2-h^2} + jh] = -\frac{\pi}{4}[\sqrt{a^2-0.1^2} + j0.1]$$

2. This is a straight line parallel to the real axis that is parameterized as a function of the input signal amplitude a and is also plotted in the following figure



3. We can also determine the limit-cycle information analytically:

$$-\frac{1}{K_{eq}(a)} = -\frac{\pi}{4}[\sqrt{a^2-0.1^2} + j0.1] = G(j\omega) = \frac{1}{j\omega(j\omega+1)} = \frac{1}{-\omega^2 + j\omega}$$

4. By solving above equations,

$$\omega^3 + \omega = \frac{40}{\pi} \approx 12.73 \qquad\qquad a^2 - 0.01 = \left(\frac{1}{\omega^2+1}\frac{4}{\pi}\right)^2$$

we can get the solutions

$$\therefore \quad \omega_l = 2.2 \quad \text{and} \quad a_l = 0.24$$