2 Statics of Open Chains

- Using the principle of conservation of power, we have

power at the joints = (power to move the robot) + (power at the end-effector)

- Considering the robot to be at static equilibrium (no power is being used to move the robot), we can equate the power at the joints to the power at the end-effector

$$\tau^T \dot{\theta} = \mathcal{F}_b^T \mathcal{V}_b = \mathcal{F}_b^T J_b(\theta)\dot{\theta} = (J_b^T(\theta)\mathcal{F}_b)^T \dot{\theta}$$

where τ is the column vector of the joint torques.

- The joint torque at the static equilibrium becomes

$$\tau = J_b^T(\theta)\mathcal{F}_b \quad \leftrightarrow \quad \tau = J_s^T(\theta)\mathcal{F}_s$$

- Independently of the choice of the frame, we have

$$\tau = J^T(\theta)\mathcal{F}$$

- If an external wrench $-\mathcal{F}$ is applied to the end-effector when the robot is at equilibrium with joint values θ, the above equation calculates the joint torques τ needed to generate the opposing wrench $\mathcal{F}$, keeping the robot at equilibrium. This is important in force control of a robot.
  - If $J^T(\theta)$ is a 6×6 invertible matrix, then clearly

$$\mathcal{F} = J^{-T}(\theta)\tau$$

  - If the robot is redundant ($n > 6$) then the robot is not immobilized and the joint torques may cause internal motions of the links. The static equilibrium assumption is no longer satisfied.
  - If $n < 6$, no matter what τ we choose, the robot cannot actively generate forces in the $6 - n$ wrench directions defined by the null space of J^T,

$$Null(J^T(\theta)) = \{ \ \mathcal{F} \mid J^T(\theta)\mathcal{F} = 0 \ \},$$

    since no actuators act in these directions.
    * The robot can, however, resist arbitrary externally applied wrenches in the space $Null(J^T(\theta))$ without moving, owing to the lack of joints that would allow motions due to these forces.
    * For example, consider a motorized rotating door with a single revolute joint ($n = 1$) and an end-effector frame at the door knob. The door can only actively generate a force at the knob that is tangential to the allowed circle of motion of the knob (defining a single direction in the wrench space), but it can resist arbitrary wrenches in the orthogonal five-dimensional wrench space without moving.

3 Singularity Analysis

- The Jacobian allows us to identify postures at which the robot's end-effector loses the ability to move instantaneously in one or more directions.
- Such a posture is called a kinematic singularity, or simply a singularity.
- Singular postures correspond to those values of θ at which the rank of $J_b(\theta)$ drops below the maximum possible value; at such postures the tip frame loses the ability to generate instantaneous spatial velocities in one or more dimensions.
- This loss of mobility at a singularity is accompanied by the ability to resist arbitrary wrenches in the direction corresponding to the lost mobility.
- The mathematical definition of a kinematic singularity is independent of the choice of body or space Jacobian.

$$\text{rank} \quad J_s(\theta) = \text{rank} \quad J_b(\theta),$$

since $Ad_{T_{sb}}$ is always full rank from $J_s = Ad_{T_{sb}}(J_b)$; in other words, singularities of the space and body Jacobian are the same.

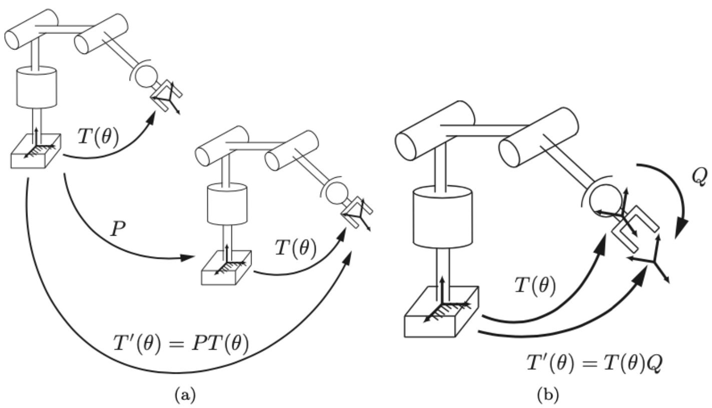

Figure 5.10: Kinematic singularities are invariant with respect to the choice of fixed and end-effector frames. (a) Choosing a different fixed frame, which is equivalent to relocating the base of the robot arm; (b) choosing a different end-effector frame.

- Kinematic singularities are also independent of the choice of fixed frame and end-effector frame.
- The FK wrt the original fixed frame is denoted $T(\theta)$, while the FK wrt the relocated fixed frame is denoted $T'(\theta) = PT(\theta)$. Since a simple calculation reveals that

$$(T')^{-1}\dot{T}' = (T^{-1}P^{-1})(P\dot{T}) = T^{-1}\dot{T} \qquad \rightarrow \qquad J_b'(\theta) = J_b(\theta)$$

  the singularities of the original and relocated robot arms are the same.
- To see that singularities are independent of the end-effector frame, suppose the FK for the original end-effector frame is given by $T(\theta)$ while the FK for the relocated end-effector frame is $T'(\theta) = T(\theta)Q$, since a simple calculation reveals that

$$\dot{T}'(T')^{-1} = (\dot{T}Q)(Q^{-1}T^{-1}) = \dot{T}T^{-1} \qquad \rightarrow \qquad J_s'(\theta) = J_s(\theta)$$

  the kinematic singularities are invariant wrt the choice of end-effector frame.

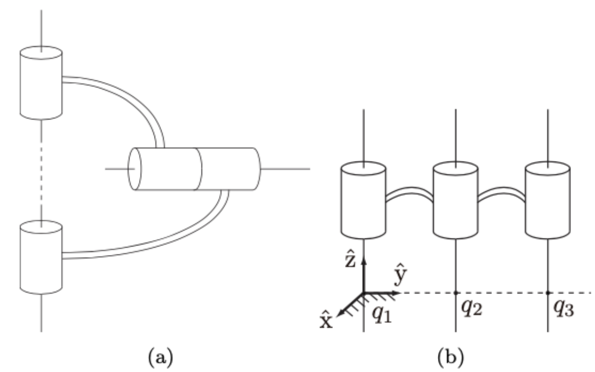

Figure 5.11: (a) A kinematic singularity in which two joint axes are collinear. (b) A kinematic singularity in which three revolute joint axes are parallel and coplanar.

Case I: Two Collinear Revolute Joint Axes

- Consider the case that two revolute joint axes are collinear. The corresponding columns of the Jacobian are

$$J_{s1}(\theta) = \begin{bmatrix} \omega_{s1} \\ \omega_{s1} \times (-q_1) \end{bmatrix} \qquad J_{s2}(\theta) = \begin{bmatrix} \omega_{s2} \\ \omega_{s2} \times (-q_2) \end{bmatrix}$$

- Since the two joint axes are collinear, we must have $\omega_{s1} = \pm\omega_{s2}$; let us assume the positive sign. Also,

$$\omega_{si} \times (q_1 - q_2) = 0 \qquad \text{for} \quad i = 1, 2$$

- Then $J_{s1} = J_{s2}$, the set $\{ J_{s1}, J_{s2}, \cdots, J_{s6} \}$ cannot be linearly independent, and the rank of $J_s(\theta)$ must be less than six.

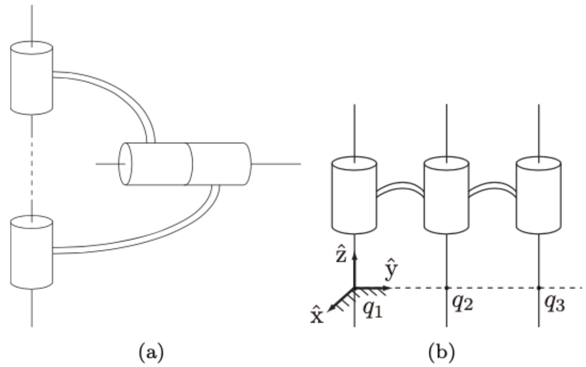

Figure 5.11: (a) A kinematic singularity in which two joint axes are collinear. (b) A kinematic singularity in which three revolute joint axes are parallel and coplanar.

Case II: Three Coplanar and Parallel Revolute Joint Axes

- Consider the case that three revolute joint axes are parallel and also lie on the same plane (three coplanar axes. The space Jacobian

$$J_s(\theta) = \begin{bmatrix} \omega_{s1} & \omega_{s1} & \omega_{s1} & \cdots \\ 0 & \omega_{s1} \times (-q_2) & \omega_{s1} \times (-q_3) & \cdots \end{bmatrix}$$

- Since q_2 and q_3 are points on the same unit axis, it is not difficult to verify that the first three columns cannot be linearly independent.

$$J_{s3} = \frac{\|q_3\|}{\|q_2\|} J_{s2} + \frac{\|q_2\| - \|q_3\|}{\|q_2\|} J_{s1}$$

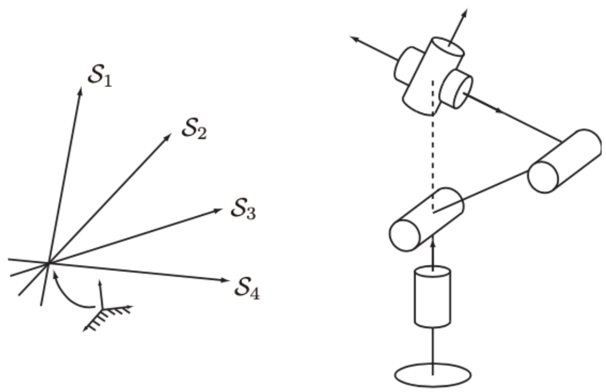

Figure 5.12: A kinematic singularity in which four revolute joint axes intersect at a common point.

Case III: Four Revolute Joint Axes Intersecting at a Common Point

- Consider the case where four revolute joint axes intersect at a common point. Since $q_1 = q_2 = q_3 = q_4 = 0$,

$$J_s(\theta) = \begin{bmatrix} \omega_{s1} & \omega_{s2} & \omega_{s3} & \omega_{s4} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

- The first four columns clearly cannot be linearly independent; one can be written as a linear combination of the other three.

$$\omega_4 = \alpha\omega_1 + \beta\omega_2 + \gamma\omega_3$$

  with some constants α, β, γ.

- Such a singularity occurs, for example, when the wrist center of an elbow-type robot arm is directly above the shoulder.

Case IV: Four Coplanar Revolute Joints

- Consider the case in which four revolute joint axes are coplanar. Choose a fixed frame such that the joint axes all lie on the $x-y$ plane; in this case the unit vector $\omega_{si} \in \Re^3$ in the direction of joint axis i is of the form

$$\omega_{si} = \begin{bmatrix} \omega_{six} \\ \omega_{siy} \\ 0 \end{bmatrix}$$

- Similarly, any reference point $q_i \in \Re^3$ lying on joint axis i is of the form

$$q_i = \begin{bmatrix} q_{ix} \\ q_{iy} \\ 0 \end{bmatrix} \qquad \rightarrow \qquad v_{si} = \omega_{si} \times (-q_i) = \begin{bmatrix} 0 \\ 0 \\ w_{siy}q_{ix} - w_{six}q_{iy} \end{bmatrix}$$

- The first four columns of the space Jacobian $J_s(\theta)$ has the following form and they cannot be linearly independent since they only have three nonzero components.

$$J_s(\theta) = \begin{bmatrix} \omega_{s1x} & \omega_{s2x} & \omega_{s3x} & \omega_{s4x} & \cdots \\ \omega_{s1y} & \omega_{s2y} & \omega_{s3y} & \omega_{s4y} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ w_{s1y}q_{1x} - w_{s1x}q_{1y} & w_{s2y}q_{2x} - w_{s2x}q_{2y} & w_{s3y}q_{3x} - w_{s3x}q_{3y} & w_{s4y}q_{4x} - w_{s4x}q_{4y} & \cdots \end{bmatrix}$$

Case V: Six Revolute Joints Intersecting a Common Line

- Consider six revolute joint axes intersecting a common line. Choose a fixed frame such that the common line lies along the z-axis, and select the intersection between this common line and joint axis i as the reference point $q_i \in \Re^3$ for axis i; each q_i is thus of the form $q_i = (0, 0, q_{iz})$, and

$$v_{si} = \omega_{si} \times q_i = \begin{bmatrix} -\omega_{siy} q_{iz} \\ \omega_{six} q_{iz} \\ 0 \end{bmatrix} \qquad \text{for} \quad i = 1, 2, \cdots, 6$$

- The space Jacobian $J_s(\theta)$ becomes

$$J_s(\theta) = \begin{bmatrix} \omega_{s1x} & \omega_{s2x} & \omega_{s3x} & \omega_{s4x} & \omega_{s5x} & \omega_{s6x} \\ \omega_{s1y} & \omega_{s2y} & \omega_{s3y} & \omega_{s4y} & \omega_{s5y} & \omega_{s6y} \\ \omega_{s1z} & \omega_{s2z} & \omega_{s3z} & \omega_{s4z} & \omega_{s5z} & \omega_{s6z} \\ -\omega_{s1y} q_{1z} & -\omega_{s2y} q_{2z} & -\omega_{s3y} q_{3z} & -\omega_{s4y} q_{4z} & -\omega_{s5y} q_{5z} & -\omega_{s6y} q_{6z} \\ \omega_{s1x} q_{1z} & \omega_{s2x} q_{2z} & \omega_{s3x} q_{3z} & \omega_{s4x} q_{4z} & \omega_{s5y} q_{5z} & \omega_{s6x} q_{6z} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is clearly singular.

4 Manipulability

- A kinematic singularity presents a binary proposition - a particular configuration is either kinematically singular or not
- It is reasonable to think about how close a nonsingular configuration is to being singular. In other words, one can even determine the directions in which the end-effector's ability to move is diminished, and to what extent.
- For a general n-joint open chain and a task space with coordinates $q \in \Re^m$, where $m \leq n$, the manipulability ellipsoid corresponds to the end-effector velocities for joint rates $\dot{\theta}$ satisfying $\|\dot{\theta}\| = 1$, a unit sphere in the n-dimensional joint-velocity space.

$$1 = \dot{\theta}^T \dot{\theta} = (J^{-1}\dot{q})^T (J^{-1}\dot{q}) = \dot{q}^T (JJ^T)^{-1} \dot{q} \qquad \rightarrow \qquad \dot{q}^T A^{-1} \dot{q} = 1$$

where $A = JJ^T$ with $A = A^T$ and $A > 0$ if J is full rank.

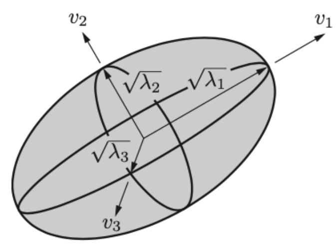

Figure 5.13: An ellipsoid visualization of $\dot{q}^{\mathrm{T}} A^{-1} \dot{q} = 1$ in the $\dot{q}$ space $\mathbb{R}^3$, where the principal semi-axis lengths are the square roots of the eigenvalues λ_i of A and the directions of the principal semi-axes are the eigenvectors v_i.

- The following equation defines an ellipsoid in the m-dimensional space.

$$\dot{q}^T A^{-1} \dot{q} = 1$$

- Letting v_i and λ_i be the eigenvectors and eigenvalues of A, the directions of the principal axes of the ellipsoid are v_i and the lengths of the principal semi-axes are $\sqrt{\lambda_i}$.

$$\dot{q}^T A^{-1} \dot{q} = 1 \quad \rightarrow \quad \dot{q}^T \left(\sum_{i=1}^{m} \lambda_i^{-1} v_i v_i^T \right) \dot{q} = \frac{1}{\lambda_1} (v_1^T \dot{q})^T (v_1^T \dot{q}) + \cdots + \frac{1}{\lambda_m} (v_m^T \dot{q})^T (v_m^T \dot{q}) = 1$$

- Furthermore, the volume V of the ellipsoid is proportional to the product of the semi-axis lengths:

$$V \text{ is proportional to } \sqrt{\lambda_1 \lambda_2 \cdots \lambda_m} = \sqrt{\det(A)} = \sqrt{\det(JJ^T)}$$

- Since six-dimensional Jacobian can be decomposed into for angular velocities and for linear velocities, we can draw each of them, $A = J_\omega J_\omega^T$ for the angular velocity manipulability ellipsoid and $A = J_v J_v^T$ for the linear velocity manipulability ellipsoid.

1st measure: ratio of the longest and shortest semi-axes of the manipulability ellipsoid

$$\mu_1(A) = \frac{\sqrt{\lambda_{max}(A)}}{\sqrt{\lambda_{min}(A)}} = \sqrt{\frac{\lambda_{max}(A)}{\lambda_{min}(A)}} \geq 1$$

where $A = JJ^T$.

- When $\mu_1(A)$ is low (i.e., close to 1) then the manipulability ellipsoid is nearly spherical or isotropic, meaning that it is equally easy to move in any direction. This situation is generally desirable.
- As the robot approaches a singularity, however, $\mu_1(A)$ goes to infinity.

2nd measure: square of $\mu_1(A)$, known as the condition number of the matrix

$$\mu_2(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)} \geq 1$$

where $A = JJ^T$.

- Smaller values (close to 1) are preferred.
- The condition number of a matrix is commonly used to characterize the sensitivity of the result of multiplying that matrix by a vector to small errors in the vector.

3rd measure: it is proportional to volume of the manipulability ellipsoid

$$\mu_3(A) = \sqrt{\lambda_1 \lambda_2 \cdots} = \sqrt{\det(A)}$$

Unlike the first two measures, a larger value is better.

Force Ellipsoid

Just like the manipulability ellipsoid, a force ellipsoid can be drawn for joint torques τ satisfying $\|\tau\| = 1$. Beginning from $\tau = J^T(\theta)\mathcal{F}$, we arrive at similar results to those above, except that now the ellipsoid satisfies

$$1 = f^T J J^T f = f^T (J J^T) f = f^T B^{-1} f$$

where $B = (JJ^T)^{-1} = A^{-1}$.

- For the force ellipsoid, the matrix B plays the same role as A in the manipulability ellipsoid.
- Since eigenvectors of any invertible matrix A are also eigenvectors of $B = A^{-1}$, the principal axes of the force ellipsoid are aligned with the principal axes of the manipulability ellipsoid.
- Furthermore, since the eigenvalues of $B = A^{-1}$ associated with each principal axis are the reciprocals of the corresponding eigenvalues of A, the lengths of the principal semi-axes of the force ellipsoid are given by $\frac{1}{\sqrt{\lambda_i}}$, where λ_i are the eigenvalues of A.
- Since the volume V_A of the manipulability ellipsoid is proportional to the product of the semi-axes, $\sqrt{\lambda_1 \lambda_2 \cdots}$, and the volume V_B of the force ellipsoid is proportional to $\frac{1}{\sqrt{\lambda_1 \lambda_2 \cdots}}$, the product of the two volumes $V_A V_B$ is constant independently of the joint variables θ.
- Therefore, positioning the robot to increase the manipulability-ellipsoid volume measure $\mu_3(A)$ simultaneously reduces the force-ellipsoid volume measure $\mu_3(B)$.
- This also explains the observation made at the start of the chapter that, as the robot approaches a singularity, V_A goes to zero while V_B goes to infinity.

5 Homework : Chapter 5

- Please solve and submit Exercise 5.1, 5.2, 5.7, 5.10, 5.11, 5.16, 5.18, 5.13, 5.21, till May 3rd (upload it as a pdf form or email me)

- If you let me know what the numbers you cannot solve until end of April, I will include the solving process in the next lecture.