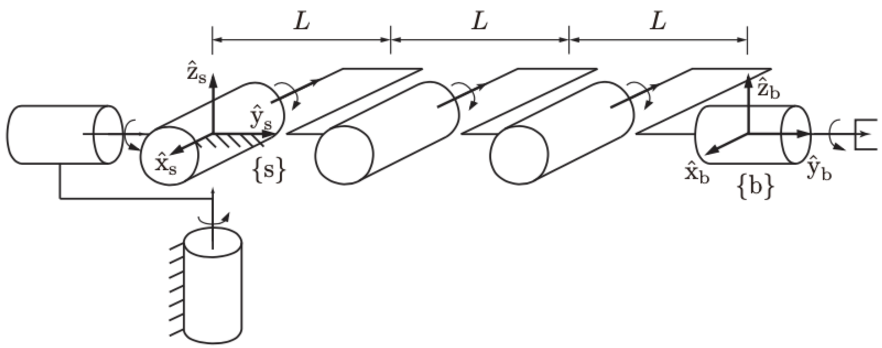

Figure 4.4: PoE forward kinematics for the 6R open chain.

Example 4.3. *(6R spatial open chain). Find the FK ?*

The forward kinematics using the space form of the PoE

$$T = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_6]\theta_6} M \quad \text{with} \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(0,1,0)	(0,0,0)
3	(-1,0,0)	(0,0,0)
4	(-1,0,0)	(0,0,L)
5	(-1,0,0)	(0,0,2L)
6	(0,1,0)	(0,0,0)

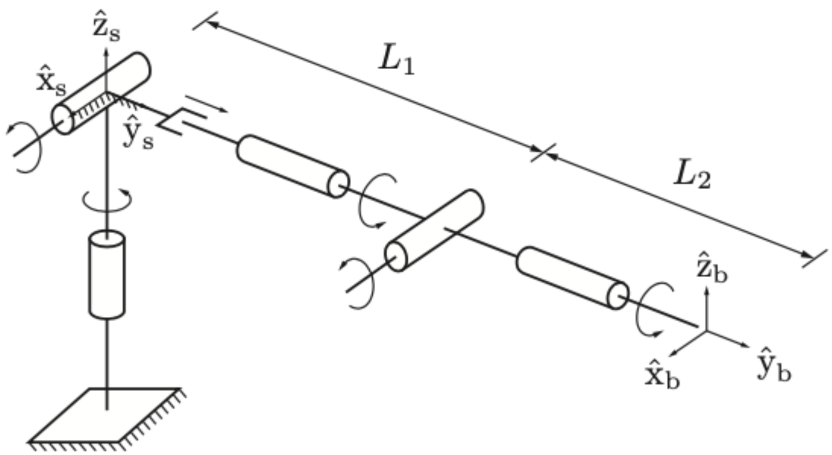

Figure 4.5: The RRPRRR spatial open chain.

Example 4.4. *(An RRPRRR spatial open chain). Find the FK ?*

The forward kinematics using the space form of the PoE

$$T = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_6]\theta_6} M \quad \text{with} \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(1,0,0)	(0,0,0)
3	(0,0,0)	(0,1,0)
4	(0,1,0)	(0,0,0)
5	(1,0,0)	$(0,0,-L_1)$
6	(0,1,0)	(0,0,0)

Note that the third joint is prismatic, so that $\omega_3 = 0$ and v_3 is a unit vector in the direction of positive translation.

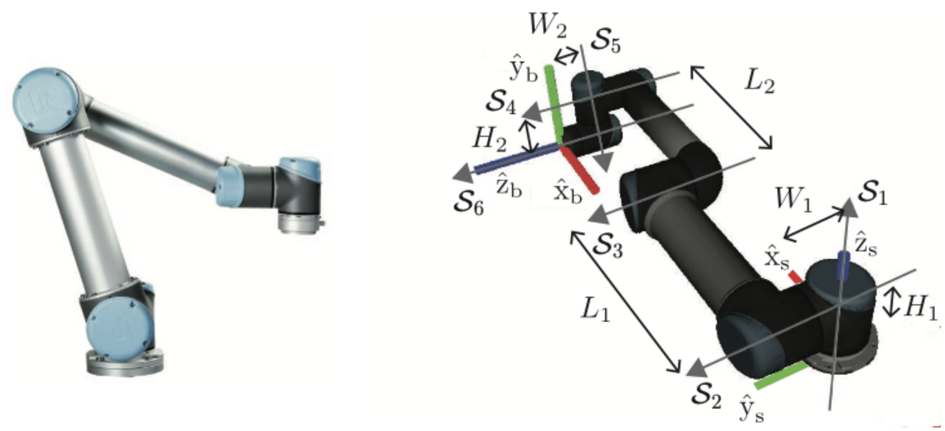

Figure 4.6: (Left) Universal Robots' UR5 6R robot arm. (Right) Shown at its zero position. Positive rotations about the axes indicated are given by the usual right-hand rule. W_1 is the distance along the $\hat{y}_s$-direction between the anti-parallel axes of joints 1 and 5. $W_1 = 109$ mm, $W_2 = 82$ mm, $L_1 = 425$ mm, $L_2 = 392$ mm, $H_1 = 89$ mm, $H_2 = 95$ mm.

Example 4.5. *(Universal Robots' UR5 6R robot arm). Find the FK at* $\theta = \left(0, -\frac{\pi}{2}, 0, 0, \frac{\pi}{2}, 0\right)$

The forward kinematics using the space form of the PoE

$$T = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_6]\theta_6} M \quad \text{with} \quad M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(0,1,0)	($-H_1$,0,0)
3	(0,1,0)	($-H_1$,0,L_1)
4	(0,1,0)	($-H_1$,0,$L_1 + L_2$)
5	(0,0,-1)	($-W_1$,$L_1 + L_2$,0)
6	(0,1,0)	($H_2 - H_1$,0,$L_1 + L_2$)

The configuration of the end-effector at $\theta = \left(0, -\frac{\pi}{2}, 0, 0, \frac{\pi}{2}, 0\right)$ is

$$\begin{aligned} T &= e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M \\ &= e^0 e^{-[\mathcal{S}_2]\frac{\pi}{2}} e^0 e^0 e^{[\mathcal{S}_5]\frac{\pi}{2}} e^0 M \qquad \text{since} \quad e^0 = I \\ &= e^{-[\mathcal{S}_2]\frac{\pi}{2}} e^{[\mathcal{S}_5]\frac{\pi}{2}} M = \begin{bmatrix} 0 & -1 & 0 & 0.095 \\ 1 & 0 & 0 & 0.109 \\ 0 & 0 & 1 & 0.988 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

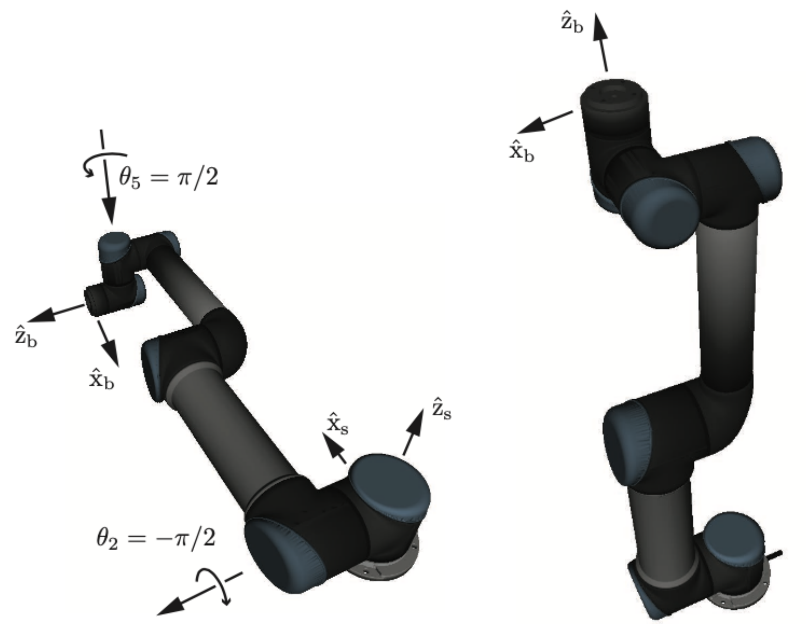

Figure 4.7: (Left) The UR5 at its home position, with the axes of joints 2 and 5 indicated. (Right) The UR5 at joint angles $\theta = (\theta_1, \ldots, \theta_6) = (0, -\pi/2, 0, 0, \pi/2, 0)$.

1.3 Second Formula: Screw Axes in End-Effector Frame (body form of PoE)

- The matrix identity can be expressed as

$$\text{if } A = M^{-1}PM \quad \text{then} \quad e^{A} = e^{M^{-1}PM} = M^{-1}e^{P}M \quad \leftrightarrow \quad Me^{M^{-1}PM} = e^{P}M$$

- Beginning with the rightmost term of the previously derived space form of PoE, if we repeatedly apply this identity, then after n iterations we obtain

$$\begin{aligned} T(\theta) &= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M \\ &= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} M e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\ &= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\ &= M e^{M^{-1}[\mathcal{S}_1]M\theta_1} \cdots e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\ &= M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n} \end{aligned}$$

where $[\mathcal{B}_i] = M^{-1}[\mathcal{S}_i]M$ i.e.,

$$\mathcal{B}_i = [Ad_{M^{-1}}]\mathcal{S}_i \qquad \text{for} \quad i = 1, 2, \cdots, n$$

- Above equation is an alternative form of the PoE formula, representing the joint axes as screw axes $\mathcal{B}_i$ in the end-effector (body) frame when the robot is at its zero position.
- It is called the body form of the PoE formula.

- In the space form,

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

  - M is first transformed by the most distal joint, progressively moving inward to more proximal joints.
  - The fixed space-frame representation of the screw axis for a more proximal joint is not affected by the joint displacement at a distal joint
  - Joint 3's displacement does not affect joint 2's screw axis representation in the space frame.

- In the body form,

$$T(\theta) = M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n}$$

  - M is first transformed by the first joint, progressively moving outward to more distal joints.
  - The body-frame representation of the screw axis for a more distal joint is not affected by the joint displacement at a proximal joint
  - Joint 2's displacement does not affect joint 3's screw axis representation in the body frame.

- Relationship bw screw axis of the space form and screw axis of the body form

$$[\mathcal{B}_i] = M^{-1}[\mathcal{S}_i]M \qquad \leftrightarrow \qquad \mathcal{B}_i = [Ad_{M^{-1}}]\mathcal{S}_i$$
$$[\mathcal{S}_i] = M[\mathcal{B}_i]M^{-1} \qquad \leftrightarrow \qquad \mathcal{S}_i = [Ad_M]\mathcal{B}_i$$

for $i = 1, 2, \cdots, n$

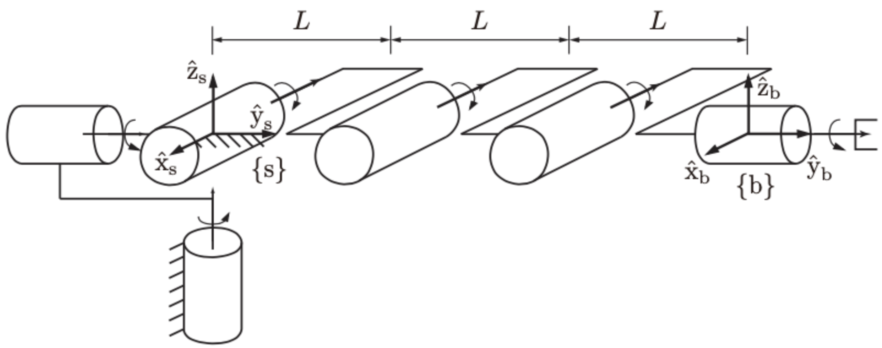

Figure 4.4: PoE forward kinematics for the 6R open chain.

Example 4.6. *(6R spatial open chain) Find the FK using the body form of the PoE ?*

The forward kinematics using the body form of the PoE

$$T = Me^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_6]\theta_6} \quad \text{with} \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

i	ω_i	v_i
1	(0,0,1)	($-3L$,0,0)
2	(0,1,0)	(0,0,0)
3	(-1,0,0)	(0,0,$-3L$)
4	(-1,0,0)	(0,0,$-2L$)
5	(-1,0,0)	(0,0,$-L$)
6	(0,1,0)	(0,0,0)

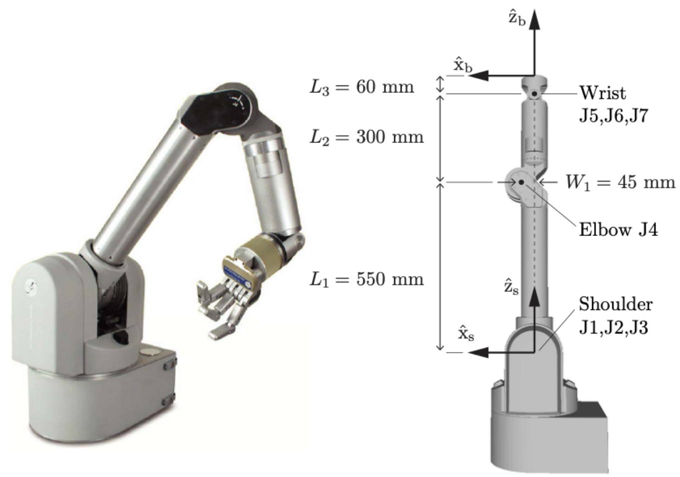

Figure 4.8: Barrett Technology's WAM 7R robot arm at its zero configuration (right). At the zero configuration, axes 1, 3, 5, and 7 are along $\hat{z}_s$ and axes 2, 4, and 6 are aligned with $\hat{y}_s$ out of the page. Positive rotations are given by the right-hand rule. Axes 1, 2, and 3 intersect at the origin of {s} and axes 5, 6, and 7 intersect at a point 60mm from {b}. The zero configuration is singular, as discussed in Section 5.3.

Example 4.7. *(Barrett WAM 7R robot arm). Find the FK using the body form of PoE at* $\theta = \left(0, \frac{\pi}{4}, 0, -\frac{\pi}{4}, 0, -\frac{\pi}{2}, 0\right)$

The forward kinematics using the body form of the PoE

$$T = Me^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_7]\theta_7} \quad \text{with} \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 + L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and}$$

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(0,1,0)	($L_1 + L_2 + L_3$,0,0)
3	(0,0,1)	(0,0,0)
4	(0,1,0)	($L_2 + L_3$,0,W_1)
5	(0,0,1)	(0,0,0)
6	(0,1,0)	(L_3,0,0)
7	(0,0,1)	(0,0,0)

The configuration at $\theta = \left(0, \frac{\pi}{4}, 0, -\frac{\pi}{4}, 0, -\frac{\pi}{2}, 0\right)$ is

$$\begin{aligned}
T &= Me^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_7]\theta_7} \\
&= Me^{0}e^{[\mathcal{B}_2]\theta_2}e^{0}e^{[\mathcal{B}_4]\theta_4}e^{0}e^{[\mathcal{B}_6]\theta_6}e^{0} \\
&= Me^{[\mathcal{B}_2]\frac{\pi}{4}}e^{-[\mathcal{B}_4]\frac{\pi}{4}}e^{-[\mathcal{B}_6]\frac{\pi}{2}} = \begin{bmatrix} 0 & 0 & -1 & 0.3157 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.6571 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

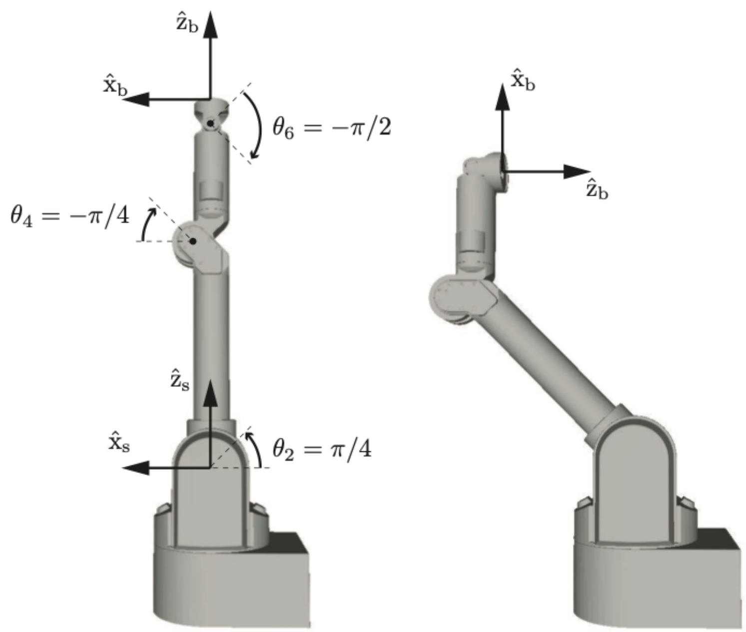

Figure 4.9: (Left) The WAM at its home configuration, with the axes of joints 2, 4, and 6 indicated. (Right) The WAM at $\theta = (\theta_1, \ldots, \theta_7) = (0, \pi/4, 0, -\pi/4, 0, -\pi/2, 0)$.

2 Homework : Chapter 4

- Please solve and submit Exercise 4.2, 4.5, 4.7, 4.8, 4.10, 4.12, 4.15, 4.18, 4.20, till April 16 (upload it as a pdf form or email me)
- If you let me know what the numbers you cannot solve until April 14, I will include the solving process in the next lecture.