제 8 장

Dynamics of Open Chains

- The motions of open-chain robots are considered taking into account the forces and torques that cause them.

- Just as a distinction was made between a robot's FK and IK, it is also customary to distinguish between a robot's forward dynamics (FD) and inverse dynamics (ID).

- The associated dynamic equations - also referred to as the equations of motion - are a set of second-order differential equations of the form. Especially, the ID is finding the joint forces and torques τ corresponding to the robot's state and a desired acceleration.

$$\text{ID:} \qquad \tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

  where $\theta \in \Re^n$ is the vector of joint variables, $\tau \in \Re^n$ is the vector of joint forces and torques, $M(\theta) \in \Re^{n\times n}$ is a symmetric positive-definite mass matrix, and $h(\theta, \dot{\theta}) \in \Re^n$ are forces that lump together centripetal, Coriolis, gravity, and friction terms that depend on θ and $\dot{\theta}$.

- The FD is determining the robot's acceleration $\ddot{\theta}$ given the state $(\theta, \dot{\theta})$ and the joint forces and torques,

$$\text{FD:} \qquad \ddot{\theta} = M^{-1}(\theta)(\tau - h(\theta, \dot{\theta}))$$

- A robot's dynamic equations are typically derived in one of two ways: by the Newton-Euler formulation for a rigid body or by the Lagrangian dynamics formulation derived from the kinetic and potential energy of the robot.

- The Lagrangian formalism is conceptually elegant and quite effective for robots with simple structures, e.g., with three or fewer degrees of freedom. The calculations can quickly become cumbersome for robots with more degrees of freedom.

- For general open chains, the Newton-Euler formulation leads to efficient recursive algorithms for both the inverse and forward dynamics that can also be assembled into closed-form analytic expressions for, e.g., the mass matrix $M(\theta)$ and the other terms in the dynamics equation.

1 Lagrangian Formulation

1.1 Basic Concepts and Motivating Examples

- The first step in the Lagrangian formulation of dynamics is to choose a set of independent (or generalized) coordinates $q \in \Re^n$ that describes the system's configuration.
- Once generalized coordinates have been chosen, these then define the generalized forces $f \in \Re^n$.
- The forces f and the coordinate rates $\dot{q}$ are dual to each other in the sense that the inner product $f^T \dot{q}$ corresponds to power.
- A Lagrangian function $\mathcal{L}(q, \dot{q})$ is then defined as the overall system's kinetic energy $\mathcal{K}(q, \dot{q})$ minus the potential energy $\mathcal{P}(q)$

$$\mathcal{L}(q, \dot{q}) = \mathcal{K}(q, \dot{q}) - \mathcal{P}(q)$$

- The equations of motion can now be expressed in terms of the Lagrangian as follows:

$$f = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q}$$

- Consider a particle of mass m constrained to move on a vertical line.
- The particle's configuration space is this vertical line, and a natural choice for a generalized coordinate is the height of the particle, which we denote by the scalar variable $x \in \Re$. Suppose that the gravitational force $\mathrm{m}g$ acts downward, and an external force f is applied upward. By Newton's second law, the equation of motion for the particle is

$$\mathrm{m}\ddot{x} = f - \mathrm{m}g$$

- Let us apply the Lagrangian formalism to derive the same result.
- The kinetic energy is $\frac{1}{2}\mathrm{m}\dot{x}^2$, the potential energy is $\mathrm{m}gx$, and the Lagrangian is

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}\mathrm{m}\dot{x}^2 - \mathrm{m}gx$$

  The equation of motion is then given by

$$f = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = \mathrm{m}\ddot{x} + \mathrm{m}g$$

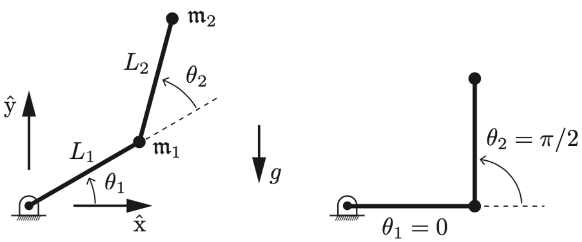

Figure 8.1: (Left) A 2R open chain under gravity. (Right) At $\theta = (0, \pi/2)$.

- Let us derive the dynamic equations for a planar 2R open chain moving in the presence of gravity.
- The chain moves in the $\hat{x}$-$\hat{y}$-plane, with gravity g acting in the $-\hat{y}$ direction.
- Two links are modeled as point masses m_1 and m_2 concentrated at the ends of each link.
- The position and velocity of the link-1 mass are then given by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} L_1 \cos\theta_1 \\ L_1 \sin\theta_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 \\ L_1 \cos\theta_1 \end{bmatrix} \dot{\theta}_1$$

while those of the link-2 mass are given by

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) \\ L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1+\theta_2) & -L_2 \sin(\theta_1+\theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) & L_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

- If the joint coordinates $\theta = (\theta_1, \theta_2)$ are chosen as the generalized coordinates, the generalized forces $\tau = (\tau_1, \tau_2)$ then correspond to joint torques (since $\tau^T\dot{\theta}$ corresponds to power).

- The Lagrangian $\mathcal{L}(\theta, \dot{\theta})$ is of the form

$$\mathcal{L}(\theta, \dot{\theta}) = \sum_{i=1}^{2} (\mathcal{K}_i - \mathcal{P}_i)$$

where the link kinetic energy terms are

$$\begin{aligned}
\mathcal{K}_1 &= \frac{1}{2}\mathrm{m}_1(\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2}\mathrm{m}_1(L_1^2\dot{\theta}_1^2 \sin^2\theta_1 + L_1^2\dot{\theta}_1^2\cos^2\theta_1) = \frac{1}{2}\mathrm{m}_1 L_1^2\dot{\theta}_1^2 \\
\mathcal{K}_2 &= \frac{1}{2}\mathrm{m}_2(\dot{x}_2^2 + \dot{y}_2^2) = \frac{1}{2}\mathrm{m}_2\left((L_1^2 + L_2^2 + 2L_1L_2\cos\theta_2)\dot{\theta}_1^2 + 2(L_2^2 + L_1L_2\cos\theta_2)\dot{\theta}_1\dot{\theta}_2 + L_2^2\dot{\theta}_2^2\right)
\end{aligned}$$

and the link potential energy terms are

$$\begin{aligned}
\mathcal{P}_1 &= \mathrm{m}_1 g y_1 = \mathrm{m}_1 g L_1 \sin\theta_1 \\
\mathcal{P}_2 &= \mathrm{m}_2 g y_2 = \mathrm{m}_2 g(L_1\sin\theta_1 + L_2\sin(\theta_1 + \theta_2))
\end{aligned}$$

- The Euler-Lagrange equations for this example are of the form

$$\begin{aligned}
\tau_1 &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} - \frac{\partial\mathcal{L}}{\partial\theta_1} \\
&= (\mathrm{m}_1 L_1^2 + \mathrm{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2))\ddot{\theta}_1 + \mathrm{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_2 - \mathrm{m}_2L_1L_2\sin\theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\
&\qquad + (\mathrm{m}_2 + \mathrm{m}_2)L_1 g\cos\theta_1 + \mathrm{m}_2L_2\cos(\theta_1 + \theta_2) \\
\tau_2 &= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} - \frac{\partial\mathcal{L}}{\partial\theta_2} \\
&= \mathrm{m}_2(L_1L_2\cos\theta_2 + L_2^2)\ddot{\theta}_1 + \mathrm{m}_2L_2^2\ddot{\theta}_2 + \mathrm{m}_2L_1L_2\dot{\theta}_1^2\sin\theta_2 + \mathrm{m}_2 g L_2\cos(\theta_1 + \theta_2)
\end{aligned}$$

- We can gather terms together into an equation of the form

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta)$$

with

$$M(\theta) = \begin{bmatrix} \mathrm{m}_1 L_1^2 + \mathrm{m}_2(L_1^2 + 2L_1 L_2 \cos\theta_2 + L_2^2) & \mathrm{m}_2(L_1 L_2 \cos\theta_2 + L_2^2) \\ \mathrm{m}_2(L_1 L_2 \cos\theta_2 + L_2^2) & \mathrm{m}_2 L_2^2 \end{bmatrix} \quad \textbf{inertia matrix}$$

$$c(\theta, \dot{\theta}) = \begin{bmatrix} -\mathrm{m}_2 L_1 L_2 \sin\theta_2 (2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \mathrm{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix} \quad \textbf{Coriolis and centrpetal torques}$$

$$g(\theta) = \begin{bmatrix} (\mathrm{m}_2 + \mathrm{m}_2) g L_1 \cos\theta_1 + \mathrm{m}_2 L_2 \cos(\theta_1 + \theta_2) \\ \mathrm{m}_2 g L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \quad \textbf{gravitational torques}$$

- These reveal that the equations of motion are linear in $\ddot{\theta}$, quadratic in $\dot{\theta}$, and trigonometric in θ.

- In (x, y) coordinates, the accelerations of the masses are written simply as second time-derivatives of the coordinates, e.g., $(\ddot{x}_2, \ddot{y}_2)$.

$$\begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -L_1\dot{\theta}_1 s_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2)s_{12} \\ L_1\dot{\theta}_1 c_1 + L_2(\dot{\theta}_1 + \dot{\theta}_2)c_{12} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \ddot{x}_2 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} -L_1\ddot{\theta}_1 s_1 - L_1\dot{\theta}_1^2 c_1 - L_2(\ddot{\theta}_1 + \ddot{\theta}_2)s_{12} - L_2(\dot{\theta}_1 + \dot{\theta}_2)^2 c_{12} \\ L_1\ddot{\theta}_1 c_1 - L_1\dot{\theta}_1^2 s_1 + L_2(\ddot{\theta}_1 + \ddot{\theta}_2)c_{12} - L_2(\dot{\theta}_1 + \dot{\theta}_2)^2 s_{12} \end{bmatrix}$$

- Accelerations are expressed as a sum of terms that are linear in the second derivatives of joint variables, $\ddot{\theta}$, and quadratic of the first derivatives of joint variables, $\dot{\theta}^T\dot{\theta}$.

- Quadratic terms containing $\dot{\theta}_i^2$ are called centripetal terms, and quadratic terms containing $\dot{\theta}_i\dot{\theta}_j$ $i \neq j$, are called Coriolis terms.

- In other words, $\ddot{\theta} = 0$ does not mean zero acceleration of the masses, due to the centripetal and Coriolis terms.

- Assume $\theta_1 = 0$, $\theta_2 = \pi/2$, i.e., $s_1 = 0$, $c_1 = 1$, $s_{12} = 1$ andd $c_{12} = 0$ and also $\ddot{\theta} = 0$, the acceleration of m_2 can be written

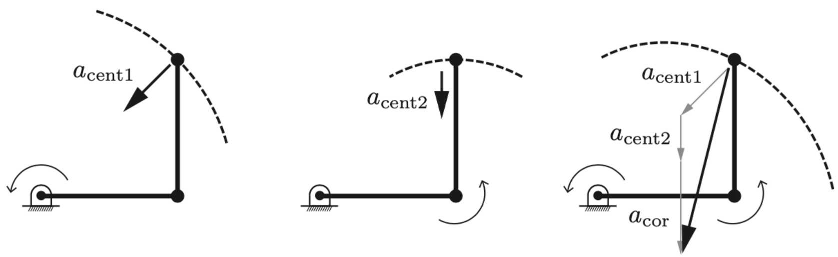

Figure 8.2: Accelerations of $\mathfrak{m}_2$ when $\theta = (0, \pi/2)$ and $\ddot{\theta} = 0$. (Left) The centripetal acceleration $a_{\text{cent1}} = (-L_1\dot{\theta}_1^2, -L_2\dot{\theta}_1^2)$ of $\mathfrak{m}_2$ when $\dot{\theta}_2 = 0$. (Middle) The centripetal acceleration $a_{\text{cent2}} = (0, -L_2\dot{\theta}_2^2)$ of $\mathfrak{m}_2$ when $\dot{\theta}_1 = 0$. (Right) When both joints are rotating with $\dot{\theta}_i > 0$, the acceleration is the vector sum of a_{cent1}, a_{cent2}, and the Coriolis acceleration $a_{\text{cor}} = (0, -2L_2\dot{\theta}_1\dot{\theta}_2)$.

$$
\begin{aligned}
\begin{bmatrix} \ddot{x}_2 \\ \ddot{y}_2 \end{bmatrix} &= \begin{bmatrix} -L_1\dot{\theta}_1^2 \\ -L_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \end{bmatrix} = \begin{bmatrix} -L_1\dot{\theta}_1^2 \\ -L_2\dot{\theta}_1^2 - L_2\dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2L_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} \\
&= \begin{bmatrix} -L_1\dot{\theta}_1^2 \\ -L_2\dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ -L_2\dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2L_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix}
\end{aligned}
$$

- Figure shows the centripetal acceleration $a_{cent1} = (-L_1\dot{\theta}_1^2, -L_2\dot{\theta}_1^2)$ when $\dot{\theta}_2 = 0$, the centripetal acceleration $a_{cent2} = (0, -L_2\dot{\theta}_2^2)$ when $\dot{\theta}_1 = 0$, and the Coriolis acceleration $a_{cor} = (0, -2L_2\dot{\theta}_1\dot{\theta}_2)$ when both $\dot{\theta}_1 > 0$ and $\dot{\theta}_2 > 0$.

1.2 General Formulation

- For general n-link open chains, the first step is to select a set of generalized coordinates $\theta \in \Re^n$ for the configuration space of the system.

- Once θ has been chosen and the generalized forces τ identified, the next step is to formulate the Lagrangian $\mathcal{L}(\theta, \dot{\theta})$

$$\mathcal{L}(\theta, \dot{\theta}) = \mathcal{K}(\theta, \dot{\theta}) - \mathcal{P}(\theta)$$

- For rigid-link robots the kinetic energy can always be written in the form

$$\mathcal{K}(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(\theta)\dot{\theta}_i\dot{\theta}_j$$

  where $m_{ij}(\theta)$ is the (i, j)th element of the $n \times n$ mass matrix $M(\theta)$

- The dynamic equations are analytically obtained by evaluating the righthand side of

$$\begin{aligned}\tau_i &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i} \qquad i = 1, 2, \cdots, n \\ &= \sum_{j=1}^{n} m_{ij}(\theta)\ddot{\theta}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_j\dot{\theta}_k + \frac{\partial \mathcal{P}}{\partial \theta_i}\end{aligned}$$

  where the $\Gamma_{ijk}(\theta)$, known as the Christoffel symbols of the first kind, are defined as follows:

$$\Gamma_{ijk}(\theta) = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial \theta_k} + \frac{\partial m_{ik}}{\partial \theta_j} - \frac{\partial m_{jk}}{\partial \theta_i}\right)$$

- This shows that the Christoffel symbols, which generate the Coriolis and centripetal terms $c(\theta, \dot{\theta})$, are derived from the mass matrix $M(\theta)$.

- By letting $g(\theta) = \frac{\partial \mathcal{P}}{\partial \theta}$, we can see explicitly that the Coriolis and centripetal terms are quadratic in the velocity by using the form

$$\tau = M(\theta)\ddot{\theta} + \dot{\theta}^T \Gamma(\theta)\dot{\theta} + g(\theta)$$

  where $\Gamma(\theta)$ is an $n \times n \times n$ matrix and the product $\dot{\theta}^T \Gamma(\theta)\dot{\theta}$ should be interpreted as follows:

$$\dot{\theta}^T \Gamma(\theta)\dot{\theta} = \begin{bmatrix} \dot{\theta}^T \Gamma_1(\theta)\dot{\theta} \\ \vdots \\ \dot{\theta}^T \Gamma_n(\theta)\dot{\theta} \end{bmatrix}$$

  where $\Gamma_i(\theta)$ is an $n \times n$ matrix with (j, k)th entry Γ_{ijk}.

- It is also common to see the dynamics written as

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta)$$

  where $C(\theta, \dot{\theta}) \in \Re^{n \times n}$ is called the Coriolis and centripetal matrix, with (i, j)th entry

$$c_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_k$$

- The Coriolis and centripetal matrix is used to prove the following passivity property (Proposition 8.1), which can be used to prove the stability of certain robot control laws.

Proposition 8.1. *The matrix* $\dot{M}(\theta) - 2C(\theta, \dot{\theta}) \in \Re^{n \times n}$ *is skew symmetric*

Proof. The (i, j)th component of $\dot{M}(\theta) - 2C(\theta, \dot{\theta})$ is

$$\begin{aligned} \dot{m}_{ij}(\theta) - 2c_{ij}(\theta, \dot{\theta}) &= \sum_{k=1}^{n} \frac{\partial m_{ij}}{\partial \theta_k}\dot{\theta}_k - \frac{\partial m_{ij}}{\partial \theta_k}\dot{\theta}_k - \frac{\partial m_{ik}}{\partial \theta_j}\dot{\theta}_k + \frac{\partial m_{jk}}{\partial \theta_i}\dot{\theta}_k \\ &= \sum_{k=1}^{n} \left(\frac{\partial m_{jk}}{\partial \theta_i} - \frac{\partial m_{ik}}{\partial \theta_j} \right) \dot{\theta}_k \end{aligned}$$

By switching the indices i and j, it can be seen that

$$\dot{m}_{ji}(\theta) - 2c_{ji}(\theta, \dot{\theta}) = -(\dot{m}_{ij}(\theta) - 2c_{ij}(\theta, \dot{\theta}))$$

thus proving that $(\dot{M}(\theta) - 2C(\theta, \dot{\theta}))^T = -(\dot{M}(\theta) - 2C(\theta, \dot{\theta}))$ as claimed. $\square$

1.3 Understanding the Mass Matrix

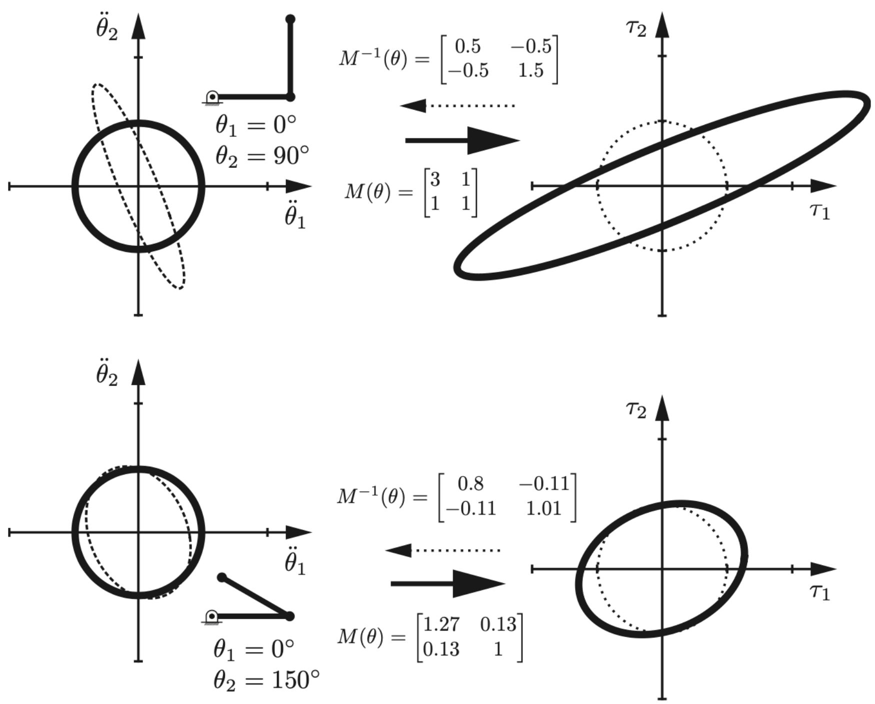

Figure 8.3: (Bold lines) A unit ball of accelerations in $\ddot{\theta}$ maps through the mass matrix $M(\theta)$ to a torque ellipsoid that depends on the configuration of the 2R arm. These torque ellipsoids may be interpreted as mass ellipsoids. The mapping is shown for two arm configurations: $(0°, 90°)$ and $(0°, 150°)$. (Dotted lines) A unit ball in τ maps through $M^{-1}(\theta)$ to an acceleration ellipsoid.

- The kinetic energy $\frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta}$ is a generalization of the familiar expression $\frac{1}{2}mv^T v$ for a point mass.
- To visualize the direction dependence of the effective mass, we can map a unit ball of joint accelerations $\{ \ \ddot{\theta} \mid \ddot{\theta}^T\ddot{\theta} = 1 \ \}$ through the mass matrix $M(\theta)$ to generate a joint force-torque ellipsoid when the mecahnism is at rest $\dot{\theta} = 0$.
- Assume $L_1 = L_2 = \mathrm{m}_1 = \mathrm{m}_2 = 1$ of 2R arm at two different joint configurations as shown in the figure.

- The torque ellipsoid can be interpreted as a direction-dependent mass ellipsoid: the same joint acceleration magnitude $\|\ddot{\theta}\|$ requires different joint torque magnitudes $\|\tau\|$ depending on the acceleration direction.
- The directions of the principal axes of the mass ellipsoid are given by the eigenvectors v_i of $M(\theta)$ and the lengths of the principal semi-axes are given by the corresponding eigenvalues λ_i.
- The acceleration $\ddot{\theta}$ is only a scalar multiple of τ when τ is along a principal axis of the ellipsoid.
- It is easier to visualize the mass matrix if it is represented as an effective mass of the end-effector, since it is possible to feel this mass directly by grabbing and moving the end-effector.
- If you grabbed the endpoint of the 2R robot, depending on the direction you applied force to it, how massy would it feel?
- Let us denote the effective mass matrix at the end-effector as $\Lambda(\theta)$, and the velocity of the end-effector as $V = (\dot{x}, \dot{y})$.
- The kinetic energy of the robot must be the same regardless of the coordinates we use, so

$$\frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} = \frac{1}{2}V^T \Lambda(\theta) V$$

- Assuming the Jacobian $J(\theta)$ satisfying $V = J(\theta)\dot{\theta}$ is invertible, the above can be rewritten as follows:

$$V^T \Lambda V = (J^{-1}V)^T M(J^{-1}V) = V^T(J^{-T} M J^{-1})V$$

- In other words, the end-effector mass matrix is

$$\Lambda(\theta) = J^{-T}(\theta) M(\theta) J^{-1}(\theta)$$

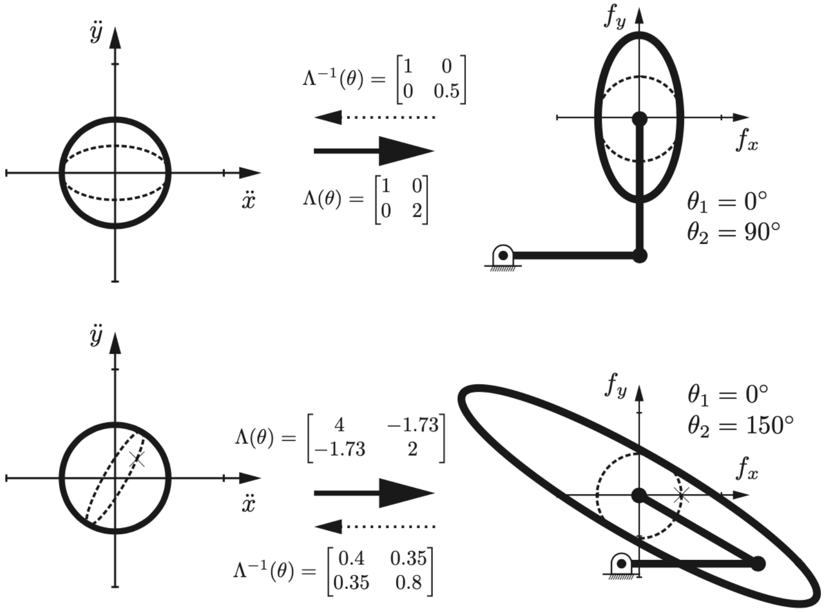

Figure 8.4: (Bold lines) A unit ball of accelerations in $(\ddot{x}, \ddot{y})$ maps through the end-effector mass matrix $\Lambda(\theta)$ to an end-effector force ellipsoid that depends on the configuration of the 2R arm. For the configuration $(\theta_1, \theta_2) = (0°, 90°)$, a force in the f_y-direction exactly feels both masses $\mathfrak{m}_1$ and $\mathfrak{m}_2$, while a force in the f_x-direction feels only $\mathfrak{m}_2$. (Dotted lines) A unit ball in f maps through $\Lambda^{-1}(\theta)$ to an acceleration ellipsoid. The $\times$ symbols for $(\theta_1, \theta_2) = (0°, 150°)$ indicate an example endpoint force $(f_x, f_y) = (1, 0)$ and its corresponding acceleration $(\ddot{x}, \ddot{y}) = (0.4, 0.35)$, showing that the force and acceleration at the endpoint are not aligned.

- Figure shows the end-effector mass ellipsoids, with principal-axis directions given by the eigenvectors of $\Lambda(\theta)$ and principal semi-axis lengths given by its eigenvalues, for the same two 2R robot configurations as in Figure 8.3.
- The endpoint acceleration $(\ddot{x}, \ddot{y})$ is a scalar multiple of the force (f_x, f_y) applied at the endpoint only if the force is along a principal axis of the ellipsoid.
- Unless $\Lambda(\theta)$ is of the form cI, where $c > 0$ is a scalar and I is the identity matrix, the mass at the endpoint feels different from a point mass. (haptic display)

1.4 Lagrangian Dynamics vs. Newton-Euler Dynamics

- Using the tools we have developed so far, the Newton-Euler formulation allows computationally efficient computer implementation, particularly for robots with many degrees of freedom, without the need for differentiation.

- The resulting equations of motion are, and must be, identical with those derived using the energy-based Lagrangian method.