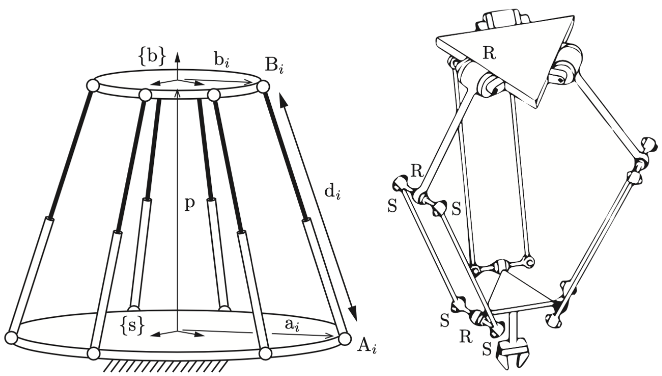
제 7 장

Kinematics of Closed Chains



(a) Stewart–Gough platform. (b) Delta robot.

Figure 7.1: Two popular parallel mechanisms.

- Closed-chain: any kinematic chain that contains one or more loops, for example, planar four-bar linkage, Stewart-Gough platform, and Delta robot.
- Parallel mechanisms: closed chains consisting of fixed and moving platforms connected by legs.
- Stewart-Gough platform is used widely as both motion simulator and six-axis force-torque sensor.
- Delta robot is a three-dof mechanism whose moving platform moves in such a way that it always remains parallel to the fixed platform. (moving parts are relatively light; very fast motions)

- For closed chains

    1. not all joints are actuated (passive joints)
    2. joint variables must satisfy many loop-closure constraint equations (constraint Jacobian)
    3. redundantly actuated or exactly actuated
    4. new types of singularities, differently from the open chains

- For general closed chains, it is usually difficult to obtain an explicit set of equations for the FK in the form $X = T(\theta)$, where $X \in SE(3)$ is the end-effector frame and $\theta \in \Re^n$ are the joint coordinates.

1 Inverse and Forward Kinematics (IK and FK)

- For serial chains, the FK is generally straightforward while the IK may be complex (e.g., there may be multiple solutions or no solution).
- For parallel mechanisms, the IK is often relatively straightforward (e.g., given the configuration of a platform, it may not be hard to determine the joint variables), while the FK may be quite complex (an arbitrarily chosen set of joint values may be infeasible or it may correspond to multiple possible configurations of the platform)

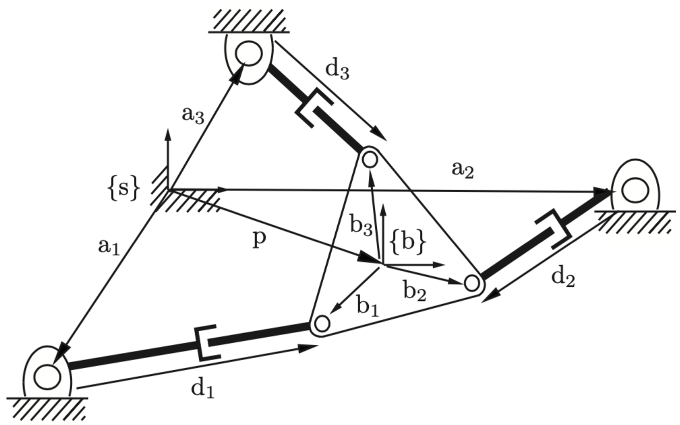


Figure 7.2: 3×RPR planar parallel mechanism.

1.1 3 x RPR Planar Parallel Mechanism

- Three prismatic joints are typically actuated while the six revolute joints are passive.
- Lengths of each of the three legs are denoted by s_i for $i = 1, 2, 3$.
- For given values of $s = (s_1, s_2, s_3)$, the FK problem is to determine the body frame's position and orientation. Conversely, the IK problem is to determine s from $T_{sb} \in SE(2)$ as follow:

$$\mathrm{d_i} = \mathrm{p} + \mathrm{b_i} - \mathrm{a_i}$$

where it is noted that $\mathrm{b_i}$ is only expressed in {b} frame and $\mathrm{a_i}, \mathrm{b_i}$ are all constant.

- Let us express above Equation in terms of {s}-frame coordinates,

$$\begin{bmatrix} d_{ix} \\ d_{iy} \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} + R_{sb} \begin{bmatrix} b_{ix} \\ b_{iy} \end{bmatrix} - \begin{bmatrix} a_{ix} \\ a_{iy} \end{bmatrix} \qquad \textbf{where} \quad R_{sb} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

- Since $s_i^2 = d_{ix}^2 + d_{iy}^2$ and

$$\begin{bmatrix} d_{ix} \\ d_{iy} \end{bmatrix} = \begin{bmatrix} p_x + b_{ix}\cos\phi - b_{iy}\sin\phi - a_{ix} \\ p_y + b_{ix}\sin\phi + b_{iy}\cos\phi - a_{iy} \end{bmatrix}$$

  we have the IK as follow:

$$s_i^2 = (p_x + b_{ix}\cos\phi - b_{iy}\sin\phi - a_{ix})^2 + (p_y + b_{ix}\sin\phi + b_{iy}\cos\phi - a_{iy})^2 \qquad \text{for } i = 1, 2, 3$$

- The FK problem of determining the body frame's position and orientation (p_x, p_y, ϕ) from the leg lengths (s_1, s_2, s_3) is not trivial.

- The following tangent half-angle substitution transforms above three equations into a system of polynomials in t, where $t = \tan\frac{\phi}{2}$, $\sin\phi = \frac{2t}{1+t^2}$, and $\cos\phi = \frac{1-t^2}{1+t^2}$

$$\begin{aligned}
s_1^2 &= \left(p_x + b_{1x}\frac{1-t^2}{1+t^2} - b_{1y}\frac{2t}{1+t^2} - a_{1x}\right)^2 + \left(p_y + b_{1x}\frac{2t}{1+t^2} + b_{1y}\frac{1-t^2}{1+t^2} - a_{1y}\right)^2 \\
s_2^2 &= \left(p_x + b_{2x}\frac{1-t^2}{1+t^2} - b_{2y}\frac{2t}{1+t^2} - a_{2x}\right)^2 + \left(p_y + b_{2x}\frac{2t}{1+t^2} + b_{2y}\frac{1-t^2}{1+t^2} - a_{2y}\right)^2 \\
s_3^2 &= \left(p_x + b_{3x}\frac{1-t^2}{1+t^2} - b_{3y}\frac{2t}{1+t^2} - a_{3x}\right)^2 + \left(p_y + b_{3x}\frac{2t}{1+t^2} + b_{3y}\frac{1-t^2}{1+t^2} - a_{3y}\right)^2
\end{aligned}$$

- After some tedious algebraic manipulation, the system of polynomials can eventually be reduced to a single sixth-order polynomial in t;

- It effectively shows that the 3 x RPR mechanism may have up to six FK solutions.

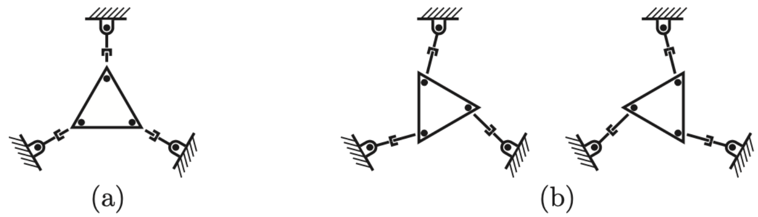
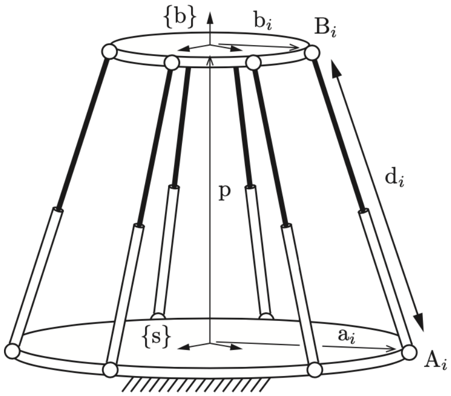


Figure 7.3: (a) The 3×RPR at a singular configuration. From this configuration, extending the legs may cause the platform to snap to a counterclockwise rotation or a clockwise rotation. (b) Two solutions to the forward kinematics when all prismatic joint extensions are identical.

- Figure (a) shows the mechanism at a singular configuration, where each leg length is identical, because extending the legs from this symmetric configuration causes the platform to rotate either clockwise or counterclockwise (bifurcation); we cannot predict it.
- Figure (b) shows two solutions to the FK when all leg lengths are identical.



(a) Stewart–Gough platform.

1.2 Stewart-Gough Platform

- The fixed and moving platforms are connected by six serial SPS structures, with the spherical joints passive and the prismatic joints actuated.
- Kinematic constraint equations

$$\mathrm{d_i} = \mathrm{p} + \mathrm{b_i} - \mathrm{a_i} \qquad \textbf{for} \quad i = 1, 2, \cdots, 6$$

$$d_i = p + Rb_i - a_i \qquad \{\text{s}\}\text{-frame coordinates}$$

- Denoting the length of leg i by s_i, we have

$$s_i^2 = d_i^T d_i = (p + Rb_i - a_i)^T (p + Rb_i - a_i)$$

- For given p and R, the six leg lengths s_i can be determined directly from the above. The IK becomes straightforward.

- For given each leg length s_i, we must solve for $p \in \Re^3$ and $R \in SO(3)$. The FK is not straightforward.
- These six constraint equations, together with six further constraints imposed by the condition $R^T R = I$, constitute a set of 12 equations in 12 unknowns (three for p, nine for R).

$$s_1^2 = d_1^T d_1 = (p + Rb_1 - a_1)^T (p + Rb_1 - a_1)$$
$$s_2^2 = d_2^T d_2 = (p + Rb_2 - a_2)^T (p + Rb_2 - a_2)$$
$$s_3^2 = d_3^T d_3 = (p + Rb_3 - a_3)^T (p + Rb_3 - a_3)$$
$$s_4^2 = d_4^T d_4 = (p + Rb_4 - a_4)^T (p + Rb_4 - a_4)$$
$$s_5^2 = d_5^T d_5 = (p + Rb_5 - a_5)^T (p + Rb_5 - a_5)$$
$$s_6^2 = d_6^T d_6 = (p + Rb_6 - a_6)^T (p + Rb_6 - a_6)$$

$$\begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad 1 = r_{11}^2 + r_{21}^2 + r_{31}^2$$

$$1 = r_{12}^2 + r_{22}^2 + r_{32}^2$$
$$1 = r_{13}^2 + r_{23}^2 + r_{33}^2$$
$$0 = r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32}$$
$$0 = r_{11}r_{13} + r_{21}r_{23} + r_{31}r_{33}$$
$$0 = r_{12}r_{13} + r_{22}r_{23} + r_{32}r_{33}$$

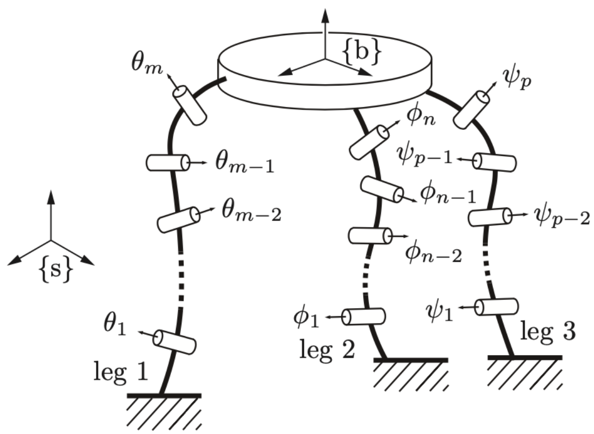


Figure 7.4: A general parallel mechanism.

1.3 General Parallel Mechanisms

- Consider a parallel mechanism, the fixed and moving platforms are connected by three open chains.
- Let the configuration of the moving platform be given by T_{sb}. Denote the FK of the three chains by $T_1(\theta), T_2(\phi)$, and $T_3(\psi)$, respectively, where $\theta \in \Re^m$, $\phi \in \Re^n$ and $\psi \in \Re^p$
- The loop-closure conditions can be written $T_{sb} = T_1(\theta) = T_2(\phi) = T_3(\psi)$. Eliminating T_{sb}, we get

$$T_1(\theta) = T_2(\phi) \qquad\qquad T_2(\phi) = T_3(\psi)$$

where each matrix equation consists of 12 equations (nine for the rotation component and three for the position component), six of which are independent: Thus there are 24 constraint equations, 12 of which are independent, with $n + m + p$ unknown variables.

- The mechanism therefore has

$$dof = n + m + p - 12$$