제 2 장

Configuration Space

- The most fundamental question about a robot is, where is it ?
- The answer is given by the robot's configuration. (A specification of the positions of all points of the robot) For example, (See Figure 2.1).
  - configuration of a door can be represented by a single number, the angle θ about its hinge
  - The configuration of a coin lying heads up on a flat table can be described by three coordinates: two coordinates (x, y) that specify the location of a particular point on the coin, and one coordinate θ that specifies the coin's orientation.

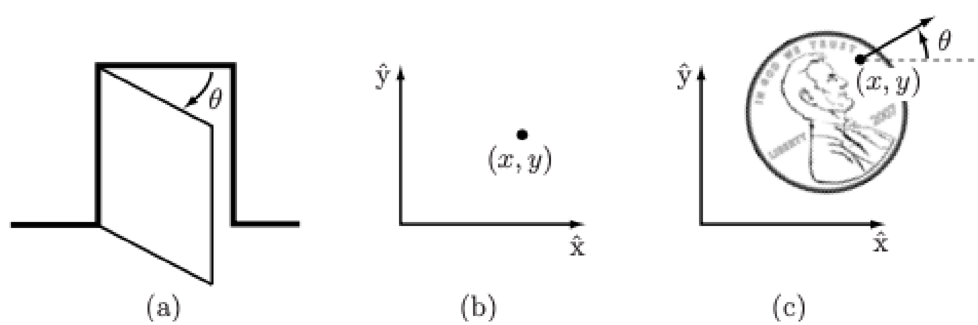


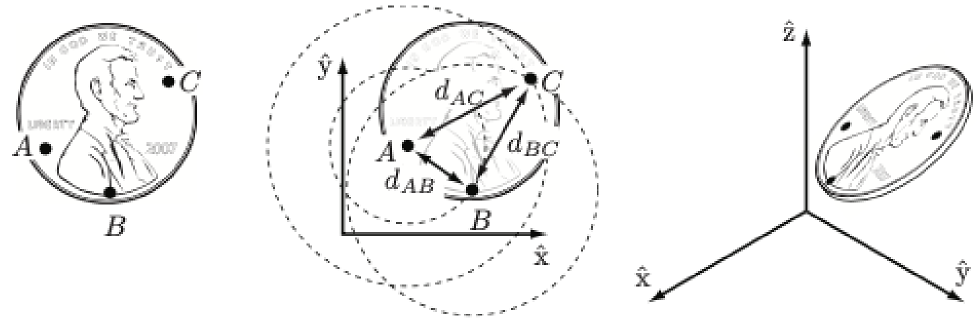
Figure 2.1: (a) The configuration of a door is described by the angle θ. (b) The configuration of a point in a plane is described by coordinates (x, y). (c) The configuration of a coin on a table is described by (x, y, θ), where θ defines the direction in which Abraham Lincoln is looking.

- The number of degrees of freedom (dof) of a robot is the smallest number of real-valued coordinates needed to represent its configuration.
  - the door has one dof
  - the coin has three dof (if we consider the heads up or tails up, then four dof)

Definition 2.1. *The* ***configuration*** *of a robot is a complete specification of the position of every point of the robot.*

- *The minimum number n of real-valued coordinates needed to represent the configuration is the number of* ***degrees of freedom (dof)*** *of the robot.*
- *The n-dimensional space containing all possible configurations of the robot is called the* ***configuration space (C-space)****.*
- *The configuration of a robot is represented by a point in its C-space.*
- ***Task space*** *means C-space of a robot's end-effector.*

1 Degrees of Freedom of a Rigid Body



(a) (b) (c)

Figure 2.2: (a) Choosing three points fixed to the coin. (b) Once the location of A is chosen, B must lie on a circle of radius d_{AB} centered at A. Once the location of B is chosen, C must lie at the intersection of circles centered at A and B. Only one of these two intersections corresponds to the "heads up" configuration. (c) The configuration of a coin in three-dimensional space is given by the three coordinates of A, two angles to the point B on the sphere of radius d_{AB} centered at A, and one angle to the point C on the circle defined by the intersection of the a sphere centered at A and a sphere centered at B.

Planar rigid body has three dof.

- See figure.(a). Once (x_A, y_A) is specified, $\rightarrow$ two dof
- the constraint $d(A, B) = d_{AB}$ restricts the choice of (x_B, y_B) to those points on the circle of radius d_{AB} centered at A. In other words, the angle ϕ_{AB} can be introduced to express the point B with the $\hat{x}$-axis. $\rightarrow$ one dof

$$d(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} = d_{AB}$$

- Once we have chosen the location of point B, the constraints $d(A, C)$ and $d(B, C)$ restrict the choice of (x_C, y_C) as point. $\rightarrow$ zero dof

Spatial rigid body has six dof.

- See figure.(c). Once (x_A, y_A, z_A) is specified, $\rightarrow$ three dof
- the constraint $d(A,B) = d_{AB}$ restricts the choice of (x_B, y_B, z_B) to those points on the sphere of radius d_{AB} centered at A. In other words, the latitude and longitude can be introduced to express the point B on the sphere. $\rightarrow$ two dof

$$d(A,B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2} = d_{AB}$$

- Once we have chosen the location of point B, the constraints $d(A,C)$ and $d(B,C)$ restrict the choice of (x_C, y_C, z_C) as one point in the intersection circle. $\rightarrow$ one dof

General rule for determining the number of dof of a system

$$\begin{aligned}
\text{dof} &= \text{sum of freedoms of the points} - \text{number of indepedent constraints} \\
&= \text{sum of variables} - \text{number of indepedent equations} \\
&= \text{sum of freedoms of the bodies} - \text{number of indepedent constraints}
\end{aligned}$$

2 Degrees of Freedom of a Robot

Consider once again the door example of Figure 2.1(a),

- without hinge joint, the door would be free to move three-dimensional space and would have six dof.
- a single rigid body (door) connected to a wall by a hinge joint has one dof by imposing five independent constraints. In other words, it leaves one independent coordinate θ
- the door's C-space is represented by some range in the interval $[0,2\pi)$ over which θ is allowed to vary.

The joints constrain the motion of the rigid body, thus reducing the overall degrees of freedom.
This observation suggests a formula for determining the number of dof of a robot, simply by counting the number of rigid bodies and joints. $\rightarrow$ Grübler's formula

2.1 Robot Joints

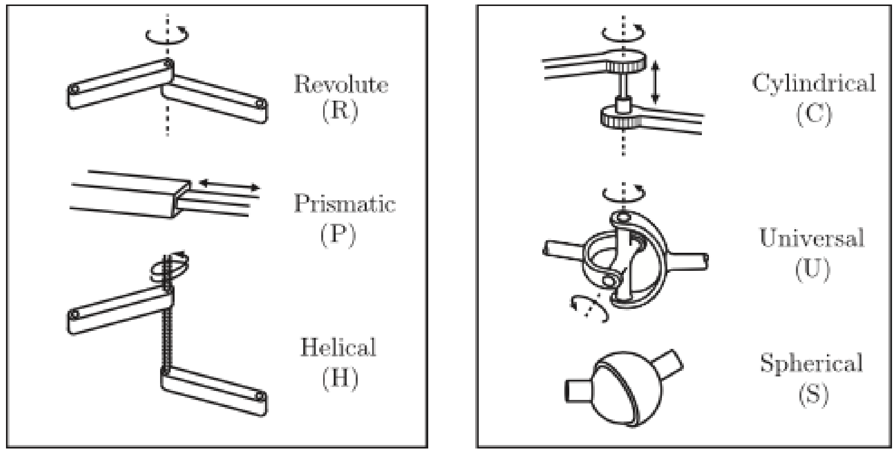


Figure 2.3: Typical robot joints.

Every joint connects exactly two links;

- The revolute joint (R) (a hinge joint) allows rotational motion about the joint axis. $\rightarrow$ one dof
- The prismatic joint (P) (a sliding or linear joint) allows translational (or rectilinear) motion along the direction of the joint axis. $\rightarrow$ one dof
- The helical joint (H) (a screw joint) allows simultaneous rotation and translation about a screw axis. $\rightarrow$ one dof
- The cylindrical joint (C) allows independent translations and rotations about a single fixed joint axis. $\rightarrow$ two dof
- The universal joint (U) consists of a pair of revolute joints arranged so that their joint axes are orthogonal. $\rightarrow$ two dof
- The spherical joint (S) (a ball-and-socket joint) functions much like our shoulder joint. $\rightarrow$ three dof

A joint can be viewed as

- providing freedoms to allow one rigid body to move relative to another
- providing constraints on the possible motions of the two rigid bodies it connects.

For example, a revolute joint can be viewed as

- allowing one freedom of motion between two rigid bodies in space,
- providing five constraints on the motion of one rigid body relative to the other.

Generalizing, the number of dof provided by the joint must equal

$$3 \text{ in the planar or } 6 \text{ in the spatial} - \text{the number of constraints provided by a joint}$$

The freedoms and constraints provided by the various joint types are summarized in Table 2.1.

Joint type	dof f	Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Helical (H)	1	N/A	5
Cylindrical (C)	2	N/A	4
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3

Table 2.1: The number of degrees of freedom f and constraints c provided by common joints.

2.2 Grübler's Formula

Proposition 2.1. *Consider a mechanism consisting of N links, where ground is regarded as a link. Let*

- *J be the number of joints,*
- *m be the number of dof of a rigid body ($m = 3$ for planar mechanisms and $m = 6$ for spatial mechanisms),*
- *f_i be the number of freedoms provided by joint i,*
- *c_i be the number of constraints provided by joint i,*
- *$f_i + c_i = m$ for all i*

Then Grübler's formula for the number of dof of the robot is

$$\begin{aligned} \textit{dof} &= m(N-1) - \sum_{i=1}^{J} c_i \\ &= m(N-1) - \sum_{i=1}^{J} (m - f_i) \\ &= m(N-1-J) + \sum_{i=1}^{J} f_i \end{aligned}$$

It is noted that

- *this formula holds only if all joint constraints are independent*
- *if they are not independent, then the formula provides a lower bound on the number of dof.*

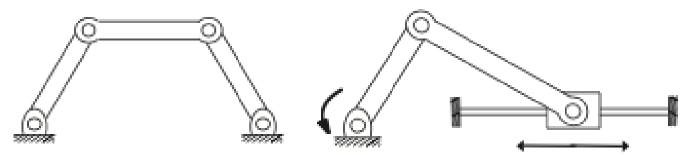


Figure 2.4: (a) Four-bar linkage. (b) Slider–crank mechanism.

Example 2.1. *(Four-bar linkage and slider-crank mechanism).*

The planar four-bar linkage shown in Figure 2.4(a) consists of four links (one of them ground) arranged in a single closed loop and connected by four revolute joints.

- For the planar mechanism, $m = 3$, the number of links $N = 4$, the number of joints $J = 4$, and the number of freedoms of the revolute joint $f_i = 1$, for $i = 1, \cdots, 4$,

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 3(4 - 1 - 4) + \sum_{i=1}^{4} 1 = 1$$

The slider-crank closed-chain mechanism of Figure 2.4(b) can be analyzed in two ways:

- the mechanism consists of three R joints and one P joint ($J = 4$ and each $f_i = 1$) and four links ($N = 4$, including the ground link)
- the mechanism consists of two R joints ($f_i = 1$) and one RP joint (the RP joint is a concatenation of a revolute and prismatic joint, so that $f_i = 2$) and three links ($N = 3$; remember that each joint connects precisely two bodies)

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 3(3 - 1 - 3) + (1 + 1 + 2) = 1$$

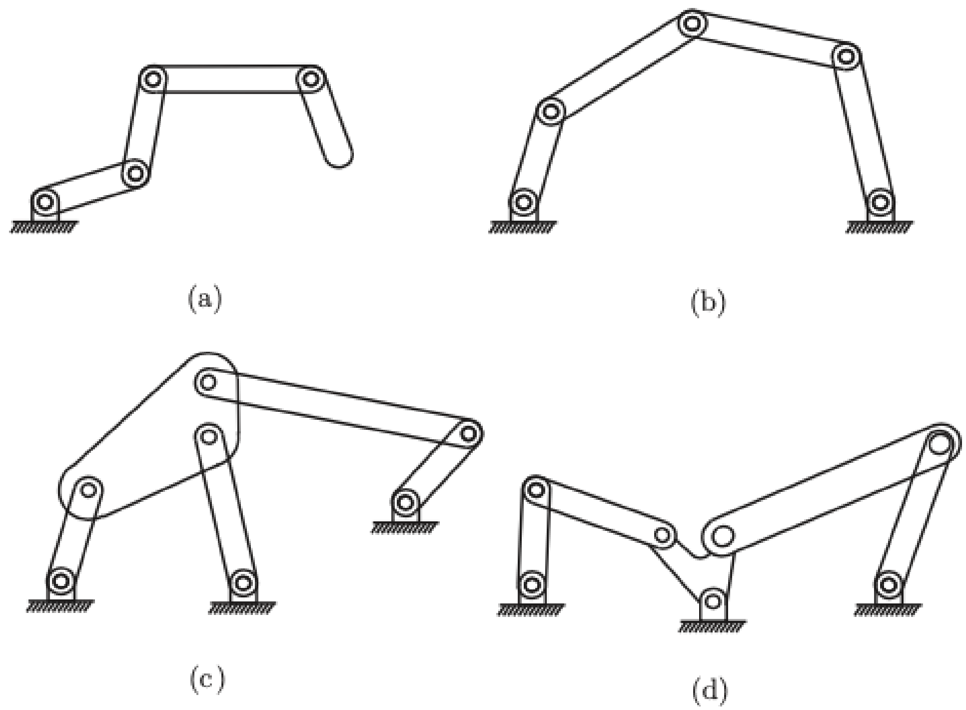


Figure 2.5: (a) k-link planar serial chain. (b) Five-bar planar linkage. (c) Stephenson six-bar linkage. (d) Watt six-bar linkage.

Example 2.2. *(Some classical planar mechanisms)*

- The k-link planar serial chain of revolute joints in Figure 2.5(a) (called a kR robot for its k revolute joints) has $N = k + 1$ links (k links plus ground), and $J = k$ joints, and, since all the joints are R, $f_i = 1$ for all i.

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 3((k + 1) - 1 - k) + k = k$$

- For the planar five-bar linkage of Figure 2.5(b), $N = 5$ (four links plus ground), $J = 5$, and since all joints are R, each $f_i = 1$.

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 3(5 - 1 - 5) + 5 = 2.$$

- For the Stephenson six-bar linkage of Figure 2.5(c), we have $N = 6$, $J = 7$, and $f_i = 1$ for all i, so that

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 3(6 - 1 - 7) + 7 = 1.$$

- For the Watt six-bar linkage of Figure 2.5(d), we have $N = 6$, $J = 7$, and $f_i = 1$ for all i,

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 3(6 - 1 - 7) + 7 = 1.$$

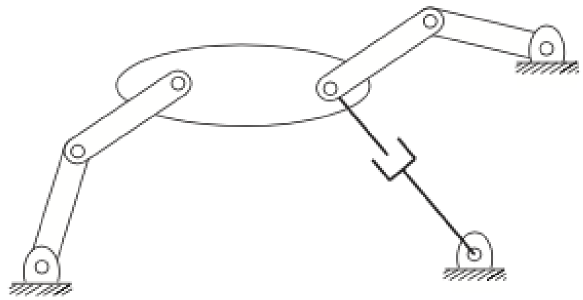


Figure 2.6: A planar mechanism with two overlapping joints.

Example 2.3. *(A planar mechanism with overlapping joints)*

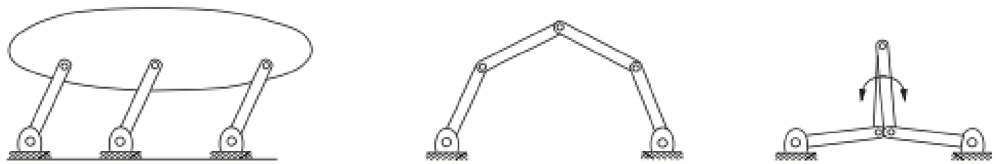
The planar mechanism illustrated in Figure 2.6 has three links that meet at a single point on the right of the large link. Recalling that a joint by definition connects exactly two links, the joint at this point of intersection should not be regarded as a single revolute joint. Rather, it is correctly interpreted as two revolute joints overlapping each other.

- The mechanism consists of eight links ($N = 8$), eight R joints, and one P joint ($J = 9$).

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 3(8 - 1 - 9) + 9(1) = 3.$$

- The lower-right revolute-prismatic joint pair can be regarded as a single two-dof (RP) joint. In this case the number of links is $N = 7$, with seven R joints, and a single two-dof RP pair ($J = 8$).

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 3(7 - 1 - 8) + 7(1) + 1(2) = 3.$$



(a) (b)

Figure 2.7: (a) A parallelogram linkage. (b) The five-bar linkage in a regular and singular configuration.

Example 2.4. *(Redundant constraints and singularities)*

- For the parallelogram linkage of Figure 2.7(a), $N = 5$, $J = 6$, and $f_i = 1$ for each joint.

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 3(5 - 1 - 6) + 6 = 0. \quad \text{rigid structure. Is it right?}$$

- Since constraints are not independent, any one of the three parallel links, with its two joints, has no effect on the motion of the mechanism, so we should remove one link

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 3(4 - 1 - 4) + 4 = 1. \quad \text{constraints are independent}$$

- For the two-dof planar five-bar linkage of Figure 2.7(b), if the two joints connected to ground are locked at some fixed angle, the five-bar linkage should then become a rigid structure.
- If the two middle links are the same length and overlap each other, as illustrated in Figure 2.7(b), these overlapping links can rotate freely about the two overlapping joints.

Grübler's formula provides a lower bound on the dof only if the constraints are not independent.

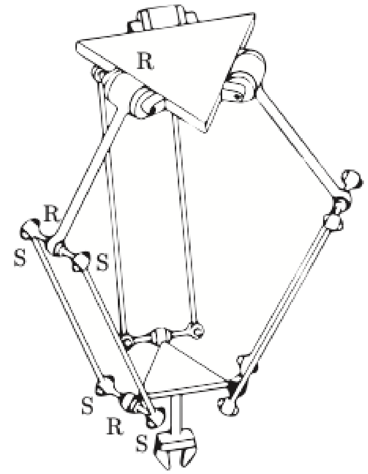


Figure 2.8: The Delta robot.

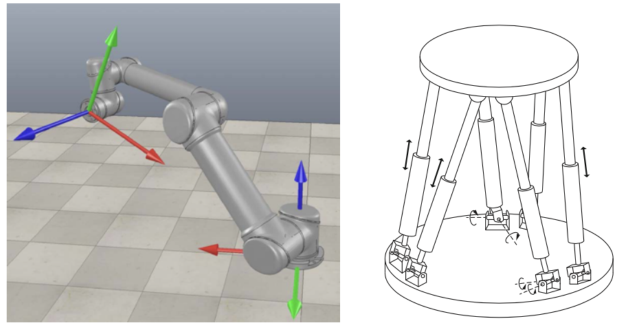
Example 2.5. *(Delta robot)*

The Delta robot of Figure 2.8 consists of two platforms - the lower one mobile, the upper one stationary - connected by three legs. Each leg contains a parallelogram closed chain and consists of three R joints, four S joints, and five links. Adding the two platforms,

- there are $N = 17$ links and $J = 21$ joints (nine R and 12 S).

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 6(17 - 1 - 21) + 9(1) + 12(3) = 15.$$

- Of these 15 degrees of freedom, however, only three are visible at the end-effector on the moving platform.
- The other 12 internal degrees of freedom are accounted for by torsion of the 12 links in the parallelograms (each of the three legs has four links in its parallelogram) about their long axes.



(a) An open-chain industrial manipulator, visualized in V-REP [154].

(b) Stewart–Gough platform. Closed loops are formed from the base platform, through the legs, through the top platform, and through the legs back to the base platform.

Figure 1.1: Open-chain and closed-chain robot mechanisms.

Example 2.6. *(Stewart-Gough platform)*

The Stewart-Gough platform of Figure 1.1(b) consists of two platforms - the lower one stationary and regarded as ground, the upper one mobile - connected by six universal-prismatic-spherical (UPS) legs.

- The number of links is $N = 14$. Six U joints (each with two dof, $f_i = 2$), six P joints (each with a single dof, $f_i = 1$), and six S joints (each with three dof, $f_i = 3$). The number of joints is $J = 18$.

$$\textbf{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 6(14 - 1 - 18) + 6(1) + 6(2) + 6(3) = 6.$$

- In some versions of the Stewart-Gough platform, if the six U joints are replaced by S joints. then

$$\textbf{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i = 6(14 - 1 - 18) + 6(1) + 6(3) + 6(3) = 12.$$

  allowing (six internal) torsional rotations about the leg axis.