(PID) 1.1 What is PID Control?

1. Since Ziegler and Nichols' PID tuning rules (1942) had been published, the PID control has survived the challenges of advanced control theories,

   - LQG control (or $\mathcal{H}_2$ control), $\mathcal{H}_\infty$ control
   - adaptive control, robust control, and so forth.

2. In PID control,

   - Proportional control : the present effort making a present state into desired state,
   - Integral control : the accumulated effort using the experience information of bygone state
   - Derivative control : the predictive effort reflecting the tendency information for ongoing state.

3. The PID control

   a) has long life force

   b) has survived many challenges of advanced control theories

   c) is the simplest and most intuitive control method

   d) has been widely accepted in industry

   e) has occupied more than 90% of control loops

   f) is easy to use

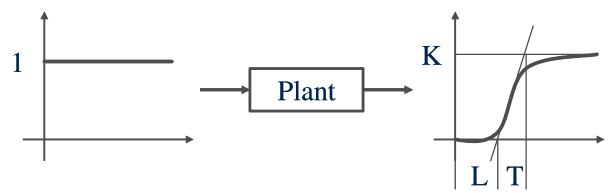
   g) has clear physical meanings

   h) can be used irrespective of system dynamics

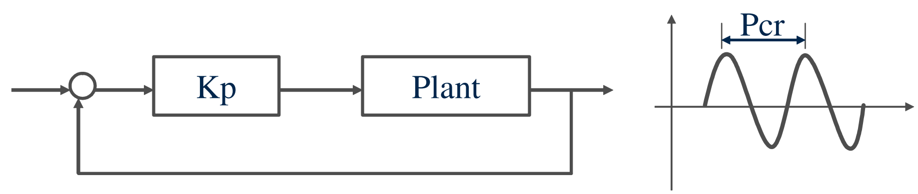
(PID) 1.2 Ziegler-Nichols Tuning Rules of PID Gains

1. Ziegler-Nichols tuning rules (1942) (characteristics) are

   - aimed at obtaining 25% maximum overshoot in step response
   - convenient when mathematical models of plants are not known
   - widely used to tune PID controllers in process control.



first method



second method

2. ZN first method: is applicable only when S-shaped curve is generated. In other words, if the plant involves neither integrator nor complex-conjugate poles, then S-shaped curve is generated.

$$PID(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) \quad \text{where} \quad K_p = 1.2\frac{T}{LK} \quad T_i = 2L \quad T_d = 0.5L$$

3. ZN second method: Using the proportional control action only, increase K_p from 0 to a critical value K_{cr} until the output first exhibits sustained oscillations (corresponding period P_{cr})

$$PID(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) \quad \text{where} \quad K_p = 0.6K_{cr} \quad T_i = 0.5P_{cr} \quad T_d = 0.125P_{cr}$$

(PID) MATLAB Example

1. Consider the following pendulum dynamics

$$ml^2\ddot{q} + mgl\sin(q) + k_f sign(\dot{q}) = \tau$$

   where m is mass, l is the length, g the gravitational acceleration constant, q the configuration, k_f the coulomb friction coefficient and τ is the control torque input.

2. Above dynamics can be expressed in terms of state-space representation by letting $x_1 \triangleq q$, $x_2 \triangleq \dot{q}$, and $u \triangleq \tau$ as follows:

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k_f}{ml^2}sign(x_2) + \frac{1}{ml^2}u = -c_1\sin x_1 - c_2 sign(x_2) + c_3 u$$

   where $c_1 = \frac{g}{l}$, $c_2 = \frac{k_f}{ml^2}$, and $c_3 = \frac{1}{ml^2}$.

3. The dynamics can be solved by using the MATLAB (filename of ‘pendulum.m’)

```
function dxdt = pendulum(t,x)
global m;
global l;
global g;
global u;
global kf;
dxdt = zeros(2,1);
dxdt(1) = x(2);
dxdt(2) = -(g/l)*sin(x(1)) -(kf/m/l/l)*sign(x(2)) + (1/m/l/l)*u;
```

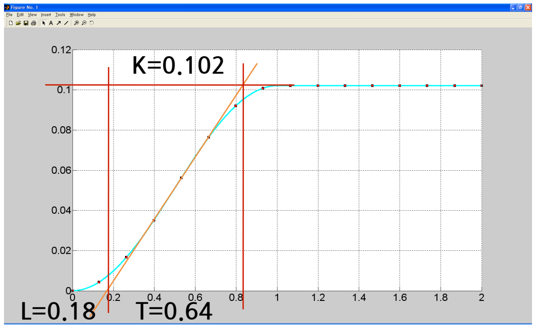
4. Main code to implement (filename of ‘ZN_first.m’)

```
close all
clear all
home

s_time = 0.002; tf = 2;
q = 0; qdot = 0; eint = 0;

global m;
global l;
global g;
global u;
global kf;

m = 1; l = 1; g = 9.806; kf = 0.5; n=1;
hold on
axis([-1.5 1.5 -1.5 1.5]);
grid
x = l*sin(q); Ax = [0, x]; y = -l*cos(q); Ay = [0, y];
p = line(Ax,Ay,'LineWidth',[5],'Color','b');

for i = 0 : s_time : tf
  u = 1;
  [t,z] = ode45('pendulum', [0, s_time], [q; qdot]);
  index = size(z); q = z(index(1), 1); qdot = z(index(1), 2);
  x = l*sin(q); Ax = [0, x]; y = -l*cos(q); Ay = [0, y];
```

```
    n=n+1;
    data(n+1,1) = i; data(n+1,2) = q;
    if rem(n,10) == 0
              set(p,'X', Ax, 'Y',Ay)
              drawnow
    end
  end
```

5. Using the following MATLAB commands

```
>> ZN_first
>> plot(data(:,1),data(:,2))
```



6. Now we can determine the gains of the PID control as follows:

$$PID(s) = \frac{U(s)}{E(s)} = K_p \left(1 + \frac{1}{T_i s} + T_d s\right) \qquad u(t) = K_p \left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d \dot{e}(t)\right)$$

where $e(t) \triangleq q_d(t) - q(t)$ and $\dot{e}(t) \triangleq \dot{q}_d(t) - \dot{q}(t)$

$$
\begin{aligned}
K_p &= 1.2\frac{T}{LK} = 1.2 \times \frac{0.64}{0.18 \times 0.102} = 41.83 \\
T_i &= 2L = 2 \times 0.18 = 0.36 \\
T_d &= 0.5L = 0.5 \times 0.18 = 0.09
\end{aligned}
$$

7. Now let us modify the 'ZN_first.m' MATLAB code instead of $u = 1$ for implementing PID control as follows:

```
qd = 90*(pi/180);
e = qd-q;
edot = 0 - qdot;
eint = eint + e*s_time;

Kp = 41.83;  Ti = 0.36; Td = 0.09;
u = Kp*(e + Td*edot + 1/Ti*eint);
```

8. Here we can confirm that the first method does not exactly show the 25% overshoot, but by adjusting the T_i and T_d a little bit, we can get the better result. Thus we can know that the first method must be a *good starting point* for PID gain tuning. For example, if we take the gains as follows, then the better result is obtained.

$$K_p = 41.83 \qquad T_i = 0.36 * 1.6 \qquad T_d = 0.09 * 1.8$$

- (HW # 5) solve 4 problems 1.6, 1.7, 1.8, and 1.9 (using 2nd Method)

(PID) 2. Nonlinear Mechanical Systems

Mechanics : equation of motion

(2.1) Lagrangian mechanics : systematic, multi-body dynamics

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q},u)}{\partial \dot{q}}\right) - \frac{\partial L(q,\dot{q},u)}{\partial q} = 0$$

where $L(q,\dot{q},u) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - P(q) + q^T u$ and $g(q) = \frac{\partial P(q)}{\partial q}$

(2.2) Hamiltonian mechanics : systematic, state-space description

$$\dot{q} = \frac{\partial H(q,p,u)}{\partial p}$$
$$\dot{p} = -\frac{\partial H(q,p,u)}{\partial q}$$

where $H(q,p,u) = p^T\dot{q} - L(q,\dot{q},u)$ and $p = \frac{\partial L(q,\dot{q},u)}{\partial \dot{q}} = M(q)\dot{q}$

(PID) 2.3 Lagrangian Control System

1. The kinetic energy of mechanical system is characterized by using Inertia matrix $M(q)$. The Lagrangian quantity is given by subtracting the potential energy from the kinetic energy plus input work-done term:

$$L(q,\dot{q},u) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - P(q) + q^T u, \qquad \text{with } q \in \Re^n \text{ and } u \in \Re^n \tag{56}$$

2. Using Lagrangian mechanics

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q},u)}{\partial \dot{q}}\right) - \frac{\partial L(q,\dot{q},u)}{\partial q} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(M(q)\dot{q}\right) - \left\{\frac{1}{2}\left[\frac{\partial}{\partial q}\left\{\dot{q}^T M(q)\right\}\right]\dot{q} - \frac{\partial P(q)}{\partial q} + u\right\} = 0$$

we have the description of Lagrangian system:

$$M(q)\ddot{q} + \left[\dot{M}(q) - \frac{1}{2}\frac{\partial}{\partial q}\left\{\dot{q}^T M(q)\right\}\right]\dot{q} + \frac{\partial P(q)}{\partial q} - u = 0,$$

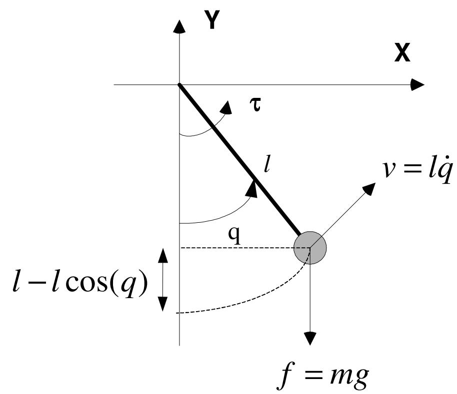
3. Here, if we define the Coriolis and centrifugal matrix and the gravitational torque/force,

$$C(q,\dot{q}) \triangleq \dot{M}(q) - \frac{1}{2}\frac{\partial}{\partial q}\left\{\dot{q}^T M(q)\right\}, \qquad g(q) \triangleq \frac{\partial P(q)}{\partial q} \qquad u \triangleq \tau$$

then we can get the Lagrangian system as following well-known equation:

$$\therefore \quad M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau \tag{57}$$

4. (Example 2.1) Obtain the Lagrangian system of the pendulum dynamics?



- The Lagrangian quantity is

$$L(q, \dot{q}, u) = K.E - P.E + qu = \frac{1}{2}m(l\dot{q})^2 - mgl[1 - \cos(q))] + qu$$

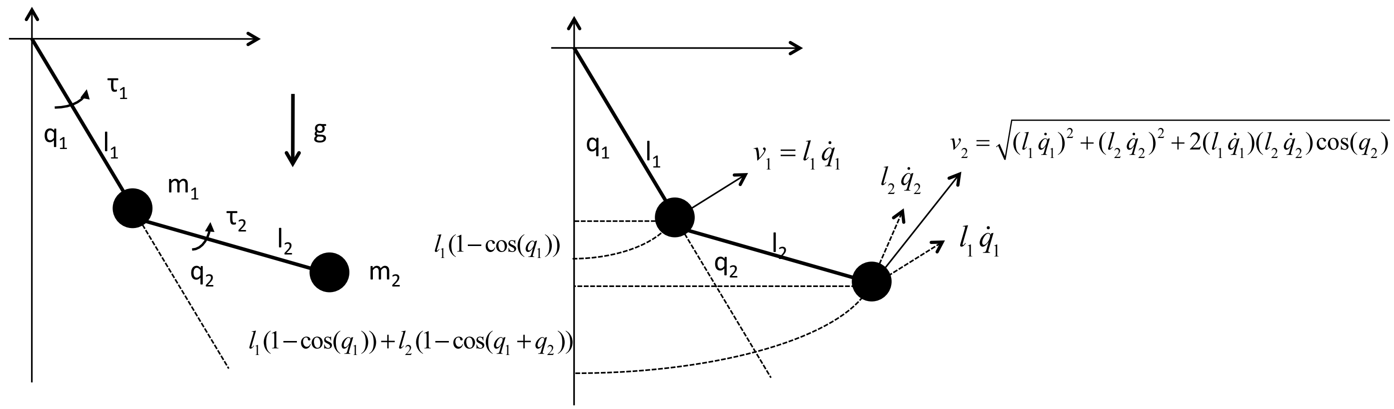
- For the given Lagrangian function, we can get the following

$$\frac{\partial L}{\partial \dot{q}} = ml^2\dot{q}$$
$$\frac{\partial L}{\partial q} = -mgl\sin(q) + u$$

- Therefore, the Lagrangian equation of motion by lettering $u = \tau$ is obtained as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad \rightarrow \quad \therefore \quad ml^2\ddot{q} + mgl\sin(q) = \tau$$

5. (Example 2.2) Obtain the Lagrangian system of two-link manipulator?



- The Lagrangian quantity is

$$K.E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1l_1^2\dot{q}_1^2 + \frac{1}{2}m_2(l_1^2\dot{q}_1^2 + l_2^2\dot{q}_2^2 + 2l_1l_2\dot{q}_1\dot{q}_2c_2)$$
$$P.E = m_1gl_1(1-c_1) + m_2g[l_1(1-c_1) + l_2(1-c_{12})]$$
$$L(q,\dot{q},u) = K.E - P.E + q_1u_1 + q_2u_2$$

where $c_1 = \cos q_1$, $s_1 = \sin q_1$, $c_2 = \cos q_2$, $s_2 = \sin q_2$, $c_{12} = \cos(q_1+q_2)$ and $s_{12} = \sin(q_1+q_2)$.

- For the given Lagrangian function, we can get the following

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{q}_1} \\ \frac{\partial L}{\partial \dot{q}_2} \end{bmatrix} = \begin{bmatrix} m_1l_1^2\dot{q}_1 + m_2l_1^2\dot{q}_1 + m_2l_1l_2\dot{q}_2c_2 \\ m_2l_2^2\dot{q}_2 + m_2l_1l_2\dot{q}_1c_2 \end{bmatrix}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \begin{bmatrix} m_1l_1^2\ddot{q}_1 + m_2l_1^2\ddot{q}_1 + m_2l_1l_2\ddot{q}_2c_2 - m_2l_1l_2\dot{q}_2s_2\dot{q}_2 \\ m_2l_2^2\ddot{q}_2 + m_2l_1l_2\ddot{q}_1c_2 - m_2l_1l_2\dot{q}_1s_2\dot{q}_2 \end{bmatrix}$$
$$\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial q_1} \\ \frac{\partial L}{\partial q_2} \end{bmatrix} = \begin{bmatrix} -m_1gl_1s_1 - m_2gl_1s_1 - m_2gl_2s_{12} + u_1 \\ -m_2l_1l_2\dot{q}_1\dot{q}_2s_2 - m_2gl_2s_{12} + u_2 \end{bmatrix}$$

- Therefore, the Lagrangian equation of motion $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$ by letting $u_1 = \tau_1$ and $u_2 = \tau_2$ is obtained as follows:

$$\begin{bmatrix} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + m_2 l_1 l_2 \ddot{q}_2 c_2 - m_2 l_1 l_2 \dot{q}_2 s_2 \dot{q}_2 + m_1 g l_1 s_1 + m_2 g l_1 s_1 + m_2 g l_2 s_{12} - u_1 \\ m_2 l_2^2 \ddot{q}_2 + m_2 l_1 l_2 \ddot{q}_1 c_2 - m_2 l_1 l_2 \dot{q}_1 s_2 \dot{q}_2 + m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 s_2 + m_2 g l_2 s_{12} - u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} m_1 l_1^2 + m_2 l_1^2 & m_2 l_1 l_2 c_2 \\ m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -m_2 l_1 l_2 \dot{q}_2^2 s_2 \\ -m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 s_2 + m_2 l_1 l_2 \dot{q}_1 \dot{q}_2 s_2 \end{bmatrix} + \begin{bmatrix} m_1 g l_1 s_1 + m_2 g l_1 s_1 + m_2 g l_2 s_{12} \\ m_2 g l_2 s_{12} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\begin{bmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 c_2 \\ m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 & -m_2 l_1 l_2 s_2 \dot{q}_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} (m_1 + m_2) g l_1 s_1 + m_2 g l_2 s_{12} \\ m_2 g l_2 s_{12} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

where

$$M(q) = \begin{bmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 c_2 \\ m_2 l_1 l_2 c_2 & m_2 l_2^2 \end{bmatrix} \tag{58}$$

$$C(q,\dot{q}) = \begin{bmatrix} 0 & -m_2 l_1 l_2 s_2 \dot{q}_2 \\ 0 & 0 \end{bmatrix} \tag{59}$$

$$g(q) = \begin{bmatrix} (m_1 + m_2) g l_1 s_1 + m_2 g l_2 s_{12} \\ m_2 g l_2 s_{12} \end{bmatrix} \tag{60}$$

[Notice] It is easily checked that $\dot{M}(q) = C(q,\dot{q}) + C^T(q,\dot{q})$ is always satisfied as shown in the following:

$$\dot{M}(q) = \begin{bmatrix} 0 & -m_2 l_1 l_2 s_2 \dot{q}_2 \\ -m_2 l_1 l_2 s_2 \dot{q}_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -m_2 l_1 l_2 s_2 \dot{q}_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2 l_1 l_2 s_2 \dot{q}_2 & 0 \end{bmatrix} = C(q,\dot{q}) + C^T(q,\dot{q})$$

(PID) 2.4 Hamiltonian Control System (Dual form of Lagrangian)

1. The Hamiltonian quantity is derived from the generalized momentum $p = M(q)\dot{q}$ as follows.

$$
\begin{aligned}
H(q,p,u) &\triangleq p^T\dot{q} - L(q,\dot{q},u) \qquad\qquad \text{by using } \dot{q} = M^{-1}(q)p \\
&= p^T M^{-1}(q)p - \frac{1}{2}p^T M^{-1}(q)p + P(q) - q^T u \\
&= \frac{1}{2}p^T M^{-1}(q)p + P(q) - q^T u.
\end{aligned}
$$

2. Let us express the Hamiltonian system (Hamiltonian control system) for a mechanical system as simple as possible. The Hamiltonian system is calculated as follows:

$$
\begin{aligned}
\dot{q} &= \frac{\partial H(q,p,u)}{\partial p} = M^{-1}(q)p \\
\dot{p} &= -\frac{\partial H(q,p,u)}{\partial q} = -\frac{1}{2}\left[p^T\frac{\partial M^{-1}(q)}{\partial q_1}|\ldots|p^T\frac{\partial M^{-1}(q)}{\partial q_n}\right]^T p - \frac{\partial P(q)}{\partial q} + u.
\end{aligned}
$$

By using $\frac{\partial M^{-1}}{\partial q_i} = -M^{-1}\frac{\partial M}{\partial q_i}M^{-1}$ from $\frac{d}{dq_i}(MM^{-1}) = \frac{d}{dq_i}I = 0$, above equation can be rewritten as

$$
\dot{p} = \frac{1}{2}\left[\dot{q}^T\frac{\partial M}{\partial q_1}|\ldots|\dot{q}^T\frac{\partial M}{\partial q_n}\right]^T \dot{q} - g(q) + u = \frac{1}{2}\left[\frac{\partial}{\partial q}\left\{\dot{q}^T M(q)\right\}\right] M^{-1}(q)p - g(q) + u.
$$

3. If we introduce the Coriolis and centrifugal matrix to above equations, then the Hamiltonian system is described with coordinates $(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$ and $u = \tau$ as follows:

$$\dot{q} = M^{-1}(q)p \tag{61}$$

$$\dot{p} = C^T(q, \dot{q})M^{-1}(q)p - g(q) + \tau. \tag{62}$$

where

$$\begin{aligned} \dot{M}(q) &= C(q, \dot{q}) + C^T(q, \dot{q}) \\ &= \left(\dot{M}(q) - \frac{1}{2}\frac{\partial}{\partial q}\left\{\dot{q}^T M(q)\right\}\right) + C^T(q, \dot{q}) \quad \rightarrow \quad C^T(q, \dot{q}) = \frac{1}{2}\frac{\partial}{\partial q}\left\{\dot{q}^T M(q)\right\} \end{aligned}$$

(PID) Several Properties on Mechanics

1. $M(q) = M^T(q) > 0$
2. $\lambda_{min}(M)I \leq M(q) \leq \lambda_{max}(M)I$ and $\lambda_{min}(M) \leq \|M(q)\| \leq \lambda_{max}(M)$
3. $\|C(q, \dot{q})\| \leq c_0\|\dot{q}\|$ and $\|C(q, \dot{q})\dot{q}\| \leq c_0\|\dot{q}\|^2$ with $c_0 > 0$
4. $\|g(q)\| \leq g_0$ with $g_0 > 0$
5. (Lemma 1) For Lagrangian system and Hamiltonian system, the following properties are always satisfied:
   - $\dot{M}(q) = C(q, \dot{q}) + C^T(q, \dot{q})$.
   - $\dot{M}(q) - 2C(q, \dot{q})$ is skew symmetric.
   - $\dot{M}(q) - 2C^T(q, \dot{q})$ is skew symmetric.

- (HW # 6) solve 4 problems 2.3, 2.4, 2.6, and 2.7

(PID) 3. Optimization for Control

1. Pontryagin's Minimum Principle
   - Generalization of the calculus variations
   - Lagrange multiplier method for constrained optimization
   - Variational approach
2. Completion of Squares
   - Heuristic approach
   - Inverse method
3. Dynamic Programming
   - Taylor series expansion
   - Principle of optimality
   - HJB equation
   - HJI equation

(PID) 3.1 Pontryagin's Minimum Principle

1. (Pontryagin's Minimum Principle) For given system and performance index to be optimized:

$$\dot{x} = f(x, u, t) \qquad J = \lim_{t\to\infty}\left[m(x(t), t) + \int_0^t g(x(\tau), u(\tau), \tau)d\tau\right] \tag{63}$$

let us define the Hamiltonian quantity with λ termed the Lagrange multiplier

$$H(x, u, \lambda, t) = g(x, u, t) + \lambda^T f(x, u, t) \tag{64}$$

and then if the minimizing control input is applied

$$H^*(x, \lambda, t) = \min_u H(x, u, \lambda, t) \qquad \Leftarrow \qquad \frac{\partial H}{\partial u} = 0 \text{ and } \frac{\partial^2 H}{\partial u^2} > 0 \tag{65}$$

then the controlled system follows the optimal trajectory as following form:

$$\dot{x} = \frac{\partial H^*(x, \lambda, t)}{\partial \lambda} \quad \text{with the prescribed initial condition } x(0) \tag{66}$$

$$\dot{\lambda} = -\frac{\partial H^*(x, \lambda, t)}{\partial x} \quad \text{with the terminal condition } \lambda(\infty) = \frac{\partial m}{\partial x}(x(\infty), \infty) \tag{67}$$

2. (Linear Version of Pontryagin's Minimum Principle, LQR (linear quadratic regulator)) For given linear system and performance index to be optimized:

$$\dot{x} = Ax + Bu \qquad J = \lim_{t\to\infty}\left[m(x(t),t) + \frac{1}{2}\int_0^t (x^TQx + u^TRu)\,d\tau\right] \tag{68}$$

where $R = R^T > 0$ and $Q = Q^T \geq 0$, let us define the Hamiltonian quantity with λ (Lagrange multiplier)

$$H(x,u,\lambda,t) = \frac{1}{2}(x^TQx + u^TRu) + \lambda^T(Ax + Bu) \tag{69}$$

and then if the minimizing control input is applied

$$H^*(x,\lambda,t) = \min_u H(x,u,\lambda) \quad \Leftarrow \quad \frac{\partial H}{\partial u} = Ru + B^T\lambda = 0 \text{ and } \frac{\partial^2 H}{\partial u^2} = R > 0 \tag{70}$$

$$= \lambda^TAx - \frac{1}{2}\lambda^TBR^{-1}B^T\lambda + \frac{1}{2}x^TQx \quad \Leftarrow \quad u = -R^{-1}B^T\lambda \tag{71}$$

then the controlled system follows the optimal trajectory as following form:

$$\dot{x} = \frac{\partial H^*(x,\lambda,t)}{\partial \lambda} = Ax - BR^{-1}B^T\lambda \tag{72}$$

$$\dot{\lambda} = -\frac{\partial H^*(x,\lambda,t)}{\partial x} = -A^T\lambda - Qx \tag{73}$$

Above two equations can be collected to make Hamiltonian matrix as follows:

$$\begin{bmatrix}\dot{x}\\ \dot{\lambda}\end{bmatrix} = \begin{bmatrix}A & -BR^{-1}B^T\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}x\\ \lambda\end{bmatrix} \tag{74}$$

To solve the Hamiltonian matrix, the sweep method ($\lambda = Px$ with $P = P^T > 0$) is utilized

$$\dot{\lambda} = \dot{P}x + P\dot{x} \tag{75}$$

$$-A^T Px - Qx = \dot{P}x + P(Ax - BR^{-1}B^T Px) \tag{76}$$

For any $x \neq 0$, the following matrix equation (called Riccati equation) should be solved

$$\therefore \qquad \dot{P} + PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{77}$$

As a result, we can get the LQR controller for given Q and R as follows:

$$\therefore \qquad u = -R^{-1}B^T Px \qquad \rightarrow \qquad \lambda = Px$$

How to determine the diagonal terms of Q and R:

$$Q_{ii} = \frac{1}{\text{maximum acceptable value of } [x_i^2]}$$

$$R_{ii} = \frac{1}{\text{maximum acceptable value of } [u_i^2]}$$

the Matlab command $P = are(A, BR^{-1}B^T, Q)$ provides a numerical solution only when $\dot{P} = 0$.

3. (Example 3.1) For given scalar unstable system and the performance index,

$$\dot{x} = x + u \qquad\qquad J = \lim_{t\to\infty}\left[V(x(t)) + \frac{1}{2}\int_0^t x(\tau)^2 + u(\tau)^2\, d\tau\right],$$

obtain both the optimal controller and the closed-loop response with $x(0) = 1$?

- Hamiltonian quantity for a given system is

$$H(x, u, \lambda) = \frac{1}{2}x^2 + \frac{1}{2}u^2 + \lambda(x + u)$$

- To find the optimal control input, let us differentiate the Hamiltonian quantity as follows:

$$\frac{\partial H}{\partial u} = u + \lambda = 0 \quad \rightarrow \quad \therefore \quad u = -\lambda$$
$$\frac{\partial^2 H}{\partial u^2} = 1 > 0 \quad \rightarrow \quad \text{the minimization is achieved when } u = -\lambda$$

- Thus the optimized Hamiltonian quantity is obtained as follow:

$$H^*(x, \lambda) = \frac{1}{2}x^2 + \frac{1}{2}\lambda^2 + \lambda(x - \lambda)$$

- Hence, the optimal trajectories can be obtained by using the Pontryagin's minimum principle as following form:

$$\dot{x} = \frac{\partial H^*(x, \lambda)}{\partial \lambda} = -\lambda + x$$
$$\dot{\lambda} = -\frac{\partial H^*(x, \lambda)}{\partial x} = -x - \lambda$$

Here, we can get the Hamiltonian matrix

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

- Let us solve the λ by introducing the unknown positive constant p as follows:

$$\lambda \triangleq px \quad \Rightarrow \quad \dot{\lambda} = p\dot{x} \quad \Rightarrow \quad -x - px = p(x - px)$$

Now we can get the following

$$p^2 - 2p - 1 = 0 \qquad \rightarrow \qquad p = 1 \pm \sqrt{2}$$

Therefore, the positive constant p can be determined as follow

$$p = 1 + \sqrt{2} \quad \Rightarrow \quad \lambda = (1 + \sqrt{2})x$$

- Finally, the optimal controller from $u = -\lambda$ is

$$\therefore \qquad u = -(1 + \sqrt{2})x$$

- The closed-loop equation can be obtained by applying the optimal controller

$$\dot{x} = x - (1 + \sqrt{2})x = -\sqrt{2}x$$

To obtain the response of above differential equation, we take the Laplace transform

$$sX(s) - x(0) = -\sqrt{2}X(s) \qquad \rightarrow \qquad (s + \sqrt{2})X(s) = x(0) = 1 \qquad \rightarrow \qquad X(s) = \frac{1}{s + \sqrt{2}}$$

Take an inverse Laplace transform to obtain the response, then

$$\therefore \quad x(t) = e^{-\sqrt{2}t} \quad \textbf{for} \;\; t \geq 0$$

(Example) Obtain the optimal controller of

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

for given performance index

$$J = \lim_{t \to \infty} \left[V(x(t)) + \frac{1}{2} \int_0^t x^T(\tau) Q x(\tau) + u(\tau)^T R u(\tau)\, d\tau \right],$$

with

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad \textbf{and} \quad R = 1 > 0$$

The optimal control can be obtained after solving Riccati equation:

$$u = -R^{-1} B^T P x$$

$$= -\begin{bmatrix} p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= -\begin{bmatrix} 10 & 2\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A^T P + P A - P B R^{-1} B^T P + Q = 0$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 100 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -p_{12}^2 + 100 & p_{11} - p_{12} p_{22} \\ p_{11} - p_{22} p_{12} & 2p_{12} - p_{22}^2 \end{bmatrix} = 0$$

$$p_{12} = 10 \quad p_{22} = 2\sqrt{5}$$

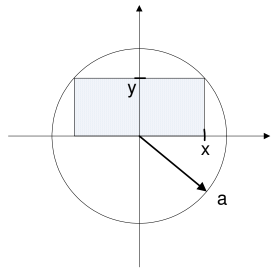
The closed-loop system is obtained as

$$\begin{aligned}\dot{x} &= (A - BR^{-1}B^TP)x \\ &= \begin{bmatrix} 0 & 1 \\ -10 & -2\sqrt{5} \end{bmatrix} x\end{aligned}$$

The characteristic equation of closed-loop system becomes

$$\begin{aligned}\det(sI - A + BR^{-1}B^TP) &= s(s + 2\sqrt{5}) + 10 \\ &= s^2 + 2\sqrt{5}s + 10 = 0 \\ &\rightarrow \quad s_{1,2} = -\sqrt{5} \pm j\sqrt{5}\end{aligned}$$

4. (Example 3.2) Determine the dimensions ($x > 0$ and $y > 0$) of the largest rectangle which can be inscribed in a semi-circle of radius a ?



$$\begin{aligned} \text{Area to be maximized} &: g(x,y) = 2xy \\ \text{Constraint equation} &: f(x,y) = x^2 + y^2 - a^2 = 0 \\ \text{Hamiltonian function} &: H(x,y,\lambda) = g(x,y) + \lambda f(x,y) \end{aligned}$$

- To solve the optimization with equality constraint equation, we firstly should obtain the Hamiltonian function.

$$H(x,y,\lambda) = g(x,y) + \lambda f(x,y) = 2xy + \lambda(x^2 + y^2 - a^2)$$

- And then we should solve the following equations

$$\begin{aligned} \frac{\partial H}{\partial x} &= 2y + \lambda(2x) = 0 \quad \rightarrow \quad \lambda = -\frac{y}{x} \\ \frac{\partial H}{\partial y} &= 2x + \lambda(2y) = 0 \quad \rightarrow \quad \lambda = -\frac{x}{y} \end{aligned}$$

So, we can get the following relation :

$$\therefore \quad x = y \quad \text{from } \lambda = -\frac{y}{x} = -\frac{x}{y}$$

- In other words, when $x = y$ and $\lambda = -1$, either minimum or maximum is achieved. And then we should confirm the definiteness of the following Hessian matrix

$$\begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial x}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial y}\right) & \frac{\partial^2 H}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2\lambda & 2 \\ 2 & 2\lambda \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \leq 0$$

- Since the Hessian matrix is negative semi-definite, the maximum is achieved when $x = y$. From the constraint equation, we can get the dimensions about x and y as follows:

$$\therefore \quad x = y = \frac{a}{\sqrt{2}} \quad \leftarrow \quad x^2 + y^2 = a^2$$