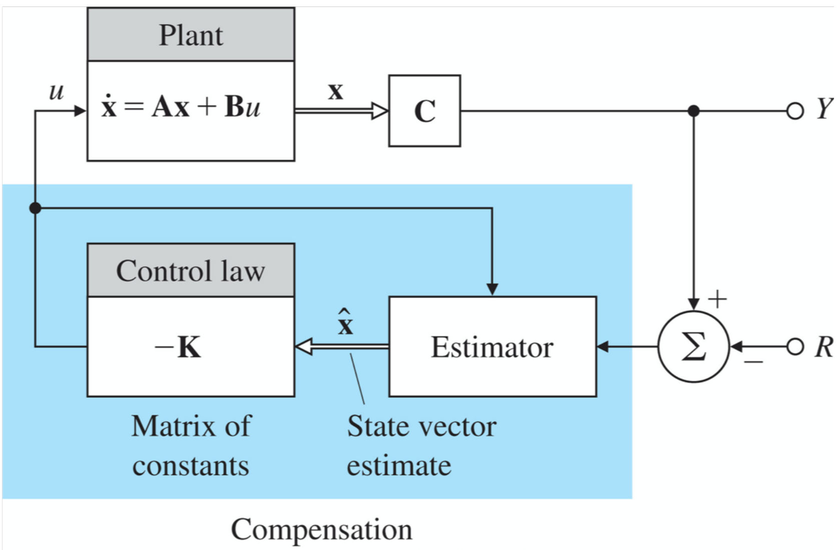
5 Control Law Design for Full-State Feedback

- The purpose of the control law is to allow us to assign a set of pole locations for the closed-loop system that will correspond to satisfactory dynamic response in terms of rise time and other measures of transient response.

- If the full state is not available, the next step is to design an estimator (sometimes called an observer), which computes an estimate of the entire state vector when provided with the measurement of the system.

- The third step consists of combining the control law and the estimator. See Fig. 7.11



- The final step is to introduce the reference input in such a way that the plant output will track external commands with acceptable rise-time, overshoot, and settling-time values.

- (7.5.1) Finding the Control Law

1. The control law is defined as feedback of a linear combination of the state variables

$$u = -Kx = -\begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

   where it is called "full-state feedback"

2. The closed-loop system under above control becomes

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= Ax - BKx \\ &= (A - BK)x \end{aligned}$$

3. The characteristic equation of the closed-loop system is

$$\det[sI - A + BK] = 0$$

4. If the desired characteristic equation is given as following form, then the required elements of K are obtained by matching both equations:

$$\alpha_c(s) = (s - s_1)(s - s_2) \cdots (s - s_n) = \det[sI - A + BK]$$

   where $s_1, s_2, \cdots, s_n$ are the desired poles locations.

5. (Example 7.14) Find the control law that places the closed-loop poles of the system so that they are both at $-2\omega_o$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Let us apply the control law

$$u = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then the desired characteristic equation should be equal to $\alpha_c(s) = (s + 2\omega_o)^2$

$$\begin{aligned} \det[sI - A + BK] &= \det \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} \right\} \\ &= s^2 + K_2 s + K_1 + \omega_o^2 = s^2 + 4\omega_o s + 4\omega_o^2 = \alpha_c(s) \end{aligned}$$

By comparing both sides, we have

$$\therefore \quad K_2 = 4\omega_o \qquad K_1 = 3\omega_o^2$$

6. Using the control canonical form (CCF): consider the following TF

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 + s^{n-2} + \cdots + a_n} \frac{X(s)}{X(s)}$$

Then we have

$$\begin{aligned}
U(s) &= (s^n + a_1 s^{n-1} + a_2 + s^{n-2} + \cdots + a_n)X(s) = s^n X(s) + a_1 s^{n-1} X(s) + \cdots + a_n X(s) \\
Y(s) &= (b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n)X(s) = b_1 s^{n-1} X(s) + \cdots + b_n X(s)
\end{aligned}$$

Let us define the states as follows:

$$\begin{aligned}
x_n &= X(s) & & \\
x_{n-1} &= sX(s) & \dot{x}_n &= x_{n-1} \\
x_{n-2} &= s^2 X(s) & \dot{x}_{n-1} &= x_{n-2} \\
&\vdots & & \\
x_2 &= s^{n-2} X(s) & \dot{x}_3 &= x_2 \\
x_1 &= s^{n-1} X(s) & \dot{x}_2 &= x_1 \\
\dot{x}_1 &= s^n X(s) = -a_1 s^{n-1} X(s) - \cdots - a_n X(s) + U(s) & \dot{x}_1 &= -a_1 x_1 - a_2 x_2 - \cdots - a_n x_n + u \\
y &= b_1 s^{n-1} X(s) + \cdots + b_n X(s) & y &= b_1 x_1 + b_2 x_2 + \cdots + b_n x_n
\end{aligned}$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} b_1 & b_2 & \cdots & \cdots & b_n \end{bmatrix} \qquad D_c = 0$$

7. Determinant of $sI - A_c + B_cK_c$ is obtained as

$$\det(sI - A_c + B_cK_c) = \det\left\{ sI - \begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 & K_3 & \cdots & K_n \end{bmatrix} \right\}$$

$$= \det \begin{bmatrix} s + a_1 + K_1 & a_2 + K_2 & \cdots & \cdots & a_n + K_n \\ -1 & s & \cdots & \cdots & 0 \\ 0 & -1 & s & \cdots & 0 \\ \vdots & & \ddots & s & \vdots \\ 0 & 0 & \cdots & -1 & s \end{bmatrix}$$

$$= s^n + (a_1 + K_1)s^{n-1} + (a_2 + K_2)s^{n-2} + \cdots + (a_n + K_n)$$

8. If the desired pole locations result in the characteristic equation given by

$$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_n$$

then the control gains can be found as follows:

$$\begin{aligned} K_1 &= \alpha_1 - a_1 \\ K_2 &= \alpha_2 - a_2 \\ &\vdots \\ K_n &= \alpha_n - a_n \end{aligned}$$

9. For given any A and B matrices, Ackermann's formula provides easy way to solve the control gain problem:

$$K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1} \alpha_c(A)$$

where the controllability matrix $\mathcal{C}$ and $\alpha_c(A)$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$
$$\alpha_c(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_n I$$

in which α_i are the coefficients of the desired characteristic polynomial.

10. (Example 7.15) Find the control law that places the closed-loop poles of the system so that they are both at $-2\omega_o$. Use Ackermann's fomula.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

First step is to construct $\alpha_c(s)$

$$\alpha_c(s) = (s + 2\omega_o)^2 = s^2 + 4\omega_o s + 4\omega_o^2$$

Second step

$$\begin{aligned} \alpha_c(A) &= A^2 + 4\omega_o A + 4\omega_o^2 I \\ &= \begin{bmatrix} -\omega_o^2 & 0 \\ 0 & -\omega_o^2 \end{bmatrix} + 4\omega_o \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{bmatrix} + 4\omega_o^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3\omega_o^2 & 4\omega_o \\ -4\omega_o^2 & 3\omega_o^2 \end{bmatrix} \end{aligned}$$

Third step is to find the controllability matrix $\mathcal{C}$

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \rightarrow \qquad \mathcal{C}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Final step is to find the gain matrix

$$\begin{aligned} \therefore \qquad K &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{C}^{-1} \alpha_c(A) \\ &= \begin{bmatrix} 3\omega_o^2 & 4\omega_o \end{bmatrix} \end{aligned}$$

11. Uncontrollable systems have certain modes, or subsystems, that are unaffected by the control. This usually means that parts of the system are physically disconnected from the input. It can be checked from the rank of the controllability matrix. In other words, the system having any uncontrollable mode loses rank of the controllability matrix. For example,

$$G(s) = \frac{s + z_o}{(s+3)(s+4)}$$

if $z_o = 3$ or 4, then the controllability matrix loses rank.

12. On the other hand, if $z_o = 2.99$, the controllability must be full, but it requires larger gain such as $K = [2052.5, -688.1]$. (It is called weakly controllable)
    - The system has to work harder and harder to achieve control as controllability slips away.
    - To move the poles a long way requires large gains.

- (Example 7.16)