- (7.6.2) Symmetric Root Locus (SRL)

1. A most effective and widely used technique of linear control system design is the optimal linear quadratic regulator (LQR).
2. The simplified version of the LQR problem is to find the control such that the performance index

$$J = \int_0^\infty \rho \frac{1}{2} z^2(t) + \frac{1}{2} u^2(t) dt$$

is minimized for the system

$$\dot{x} = Ax + Bu \qquad\qquad z = C_1 x$$

where ρ is a weighting factor of the designer's choice. The parameter ρ weighs the relative cost of z^2 with respect to the control effort u^2 in the performance index equation.

3. How to solve the optimization with constraint (Lagrange multiplier method)

a) Hamiltonian

$$H = \frac{\rho}{2} z^2(t) + \frac{1}{2} u^2(t) + \lambda^T (Ax + Bu) = \frac{\rho}{2} x^T C_1^T C_1 x + \frac{1}{2} u^2 + \lambda^T (Ax + Bu)$$

b) Optimal control input

$$\frac{\partial H}{\partial x} = \rho C_1^T C_1 x + A^T \lambda = -\dot{\lambda}$$
$$\frac{\partial H}{\partial \lambda} = Ax + Bu = \dot{x}$$
$$\frac{\partial H}{\partial u} = u + B^T \lambda = 0 \qquad \rightarrow \qquad u = -B^T \lambda$$

c) By letting $\lambda = Px$ and $\dot{\lambda} = P\dot{x}$

$$\frac{\partial H}{\partial x} = \rho C_1^T C_1 x + A^T P x = -P\dot{x} = -P(Ax + Bu) \qquad \rightarrow \qquad A^T P + PA - PBB^T P + \rho C_1^T C_1 = 0$$
$$\frac{\partial H}{\partial u} = u + B^T \lambda^T = 0 \qquad \rightarrow \qquad u = -B^T P x$$

d) Rearranging them, we have

$$u = -B^T P x \qquad \text{after solving} \qquad A^T P + PA - PBB^T P + \rho C_1^T C_1 = 0$$

where $P = P^T > 0$

4. A remarkable fact is that the control law that minimizes J is given by linear state-feedback

$$u = -Kx \qquad \text{be letting} \qquad K = B^T P$$

Here the optimal value K places the closed-loop poles at the stable roots of the symmetric root-locus (SRL) equation:

$$1 + \rho G_0(-s)G_0(s) = 0$$

where G_0 is the open-loop TF from u to z:

$$G_0(s) = \frac{Z(s)}{U(s)} = C_1(sI - A)^{-1}B = \frac{N(s)}{D(s)}$$

In other words, we can write the SRL equation in the standard root-locus form

$$1 + \rho \frac{N(-s)N(s)}{D(-s)D(s)} = 0$$

1) obtain the locus poles and zeros by reflecting the open-loop poles and zeros of the TF from U to Z across the imaginary axis, and then
2) sketch the locus.

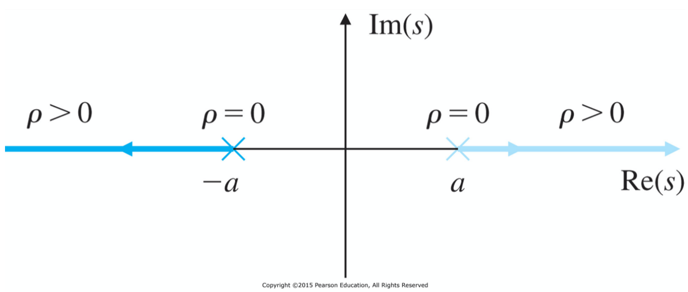
5. (Example 7.20) Plot the SRL for the following servo speed control system with $z = y$:

$$\dot{y} = -ay + u \qquad \rightarrow \qquad G_0(s) = \frac{1}{s+a}$$

The SRL equation for this example is

$$1 + \rho G_0(-s)G_0(s) = 1 + \rho \frac{1}{-s+a}\frac{1}{s+a} = 0 \qquad \rightarrow \qquad a^2 - s^2 + \rho = 0 \qquad \rightarrow \qquad s = \pm\sqrt{a^2+\rho}$$

The SRL is shown in Fig. 7.20 and the optimal (stable) pole can be determined explicitly in this case as



$$s = -\sqrt{a^2+\rho}.$$

For this closed-loop pole, the controller should be

$$u = -(\sqrt{a^2+\rho} - a)y \qquad \rightarrow \qquad \dot{y} = -(\sqrt{a^2+\rho})y$$

(LQR) For given system $\dot{y} = -ay + u$ with $z = y$

$$\dot{y} = -ay + u$$
$$z = y$$

the optimal control based on Lagrange multiplier method can be designed as follows:

$$\begin{aligned} u &= -B^T P x & A^T P + PA - PBB^T P + \rho C_1^T C_1 &= 0 \\ &= -py & -ap + p(-a) - p^2 + \rho &= 0 \\ &= -(\sqrt{a^2 + \rho} - a)y & p = -a \pm \sqrt{a^2 + \rho} & \quad \text{(positive } p \text{ is chosen)} \end{aligned}$$

The closed-loop system is obtained as

$$\dot{y} = -ay - (\sqrt{a^2 + \rho} - a)y = -(\sqrt{a^2 + \rho})y$$

It is noted that the result of LQR is same with that of the SRL.

6. (Example 7.21) Draw the SRL for the satellite system with $z = y$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad z = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

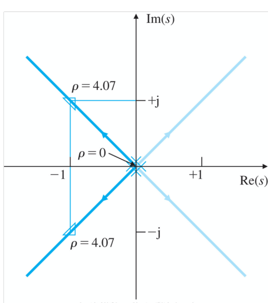
The TF can be obtained as

$$G_0(s) = \frac{1}{s^2}$$

The SRL equation for this example is

$$1 + \rho G_0(-s)G_0(s) = 1 + \rho \frac{1}{s^2}\frac{1}{s^2} = 0 \quad \rightarrow \quad s^4 + \rho = 0 \quad \rightarrow \quad s = \sqrt[4]{\rho}\left(\pm\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}\right)$$

The SRL is shown in Fig. 7.21 and the optimal (stable) poles can be determined explicitly



$$s_{1,2} = \sqrt[4]{\rho}\left(-\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}\right)$$

If $\rho = 4.07$, we have $s_{1,2} = -1 \pm j1$.

(LQR) The optimal control can be obtained as follows:

$$u = -B^T P x \qquad A^T P + PA - PBB^T P + \rho C_1^T C_1 = 0$$

$$= -\begin{bmatrix} p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} -p_{12}^2 + \rho & p_{11} - p_{12}p_{22} \\ p_{11} - p_{22}p_{12} & 2p_{12} - p_{22}^2 \end{bmatrix} = 0$$

$$= -\begin{bmatrix} \sqrt{\rho} & \sqrt{2}\sqrt[4]{\rho} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad p_{12} = \sqrt{\rho} \quad p_{22} = \sqrt{2}\sqrt[4]{\rho}$$

The closed-loop system is obtained as

$$\begin{aligned} \dot{x} &= (A - BB^T P)x \\ &= \begin{bmatrix} 0 & 1 \\ -\sqrt{\rho} & -\sqrt{2}\sqrt[4]{\rho} \end{bmatrix} x \end{aligned}$$

The characteristic equation of closed-loop system becomes

$$\det(sI - A + BB^T P) = s(s + \sqrt{2}\sqrt[4]{\rho}) + \sqrt{\rho} = s^2 + \sqrt{2}\sqrt[4]{\rho}s + \sqrt{\rho} = 0 \quad \rightarrow \quad s_{1,2} = \sqrt[4]{\rho}\left(-\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}\right)$$

It is noted that the result of LQR is same with that of the SRL.

7. (Example 7.22) Draw the SRL for the linearized equations of the simple inverted pendulum

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \qquad z = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

The TF can be obtained as

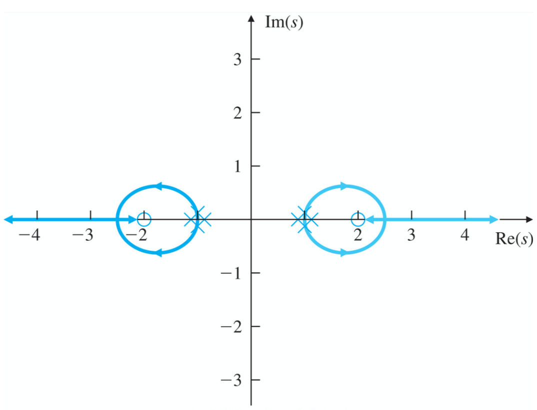
$$G_0(s) = -\frac{s+2}{s^2-1}$$

The SRL equation for this example is

$$1 + \rho G_0(-s)G_0(s) = 1 + \rho \frac{-s+2}{s^2-1}\frac{s+2}{s^2-1} = 0 \quad \rightarrow \quad (s^2-1)^2 + \rho(4-s^2) = 0 \quad \rightarrow$$

$$s^4 - (2+\rho)s^2 + (1+4\rho) = 0 \quad \rightarrow \quad s^2 = \frac{(2+\rho) \pm \sqrt{(2+\rho)^2 - 4(1+4\rho)}}{2}$$

SRL is shown in Fig. 7.24. If $\rho = 1$, we have stable closed-loop poles of $s_{1,2} = -1.36 \pm j0.606$.



8. The simplified version of LQR problem is to find the control such that performance index

$$J = \int_0^\infty \frac{1}{2}x^T Qx + \frac{1}{2}u^T Rudt$$

is minimized for the system

$$\dot{x} = Ax + Bu.$$

How to solve the optimization with constraint (Lagrange multiplier method)

a) Hamiltonian

$$H = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru + \lambda^T (Ax + Bu)$$

b) Optimal control input

$$\frac{\partial H}{\partial x} = Qx + A^T \lambda = -\dot{\lambda}$$
$$\frac{\partial H}{\partial \lambda} = Ax + Bu = \dot{x}$$
$$\frac{\partial H}{\partial u} = Ru + B^T \lambda = 0 \quad \rightarrow \quad u = -R^{-1}B^T \lambda$$

c) By letting $\lambda = Px$ and $\dot{\lambda} = P\dot{x}$

$$\frac{\partial H}{\partial x} = Qx + A^T Px = -P\dot{x} = -P(Ax + Bu) \quad \rightarrow \quad A^T P + PA - PBR^{-1}B^T P + Q = 0$$
$$\frac{\partial H}{\partial u} = Ru + B^T \lambda^T = 0 \quad \rightarrow \quad u = -R^{-1}B^T Px$$

d) Rearranging them, we have

$$u = -Kx \qquad \text{with} \qquad K = R^{-1}B^TP \qquad \text{after solving} \qquad A^TP + PA - PBR^{-1}B^TP + Q = 0$$

where

$$Q_{ii} = \text{1/maximum acceptable value of } [x_i^2]$$
$$R_{ii} = \text{1/maximum acceptable value of } [u_i^2]$$

e) MATLAB function, K = lqr(A,B,Q,R),

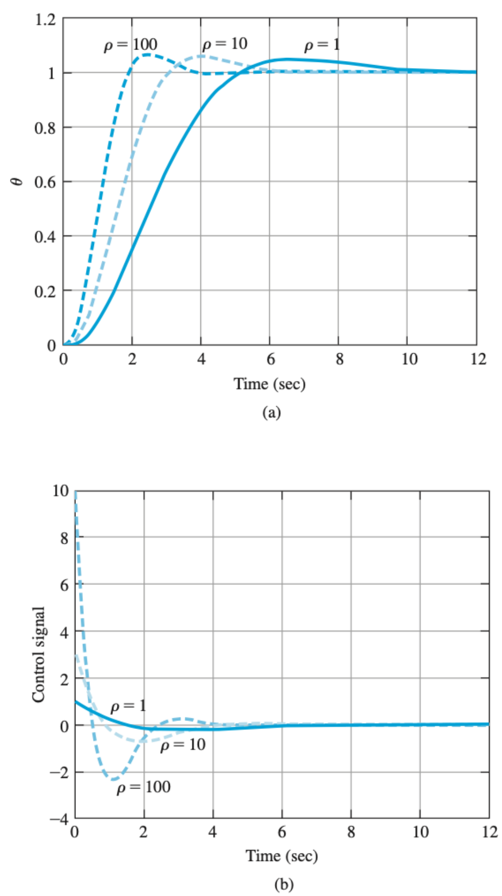
f) It is noted that $Q = \rho C_1^T C_1$ and $R = 1$ in the SRL cases.

9. Limiting behavior of LQR Regulator Poles (See Fig. 7.26)

$$J = \int_0^\infty \rho \frac{1}{2} z^2(t) + \frac{1}{2} u^2(t) dt$$

- Expensive control ($\rho \to 0$) : It penalizes the use of control energy. If the control is expensive, the optimal control does not move any of the open-loop poles except for those that are in the RHP
- Cheap control ($\rho \to \infty$) : Arbitrary control effort may be used by the optimal control law.

Figure 7.26
(a) Step responses of drone for LQR designs
(b) Control efforts for drone designs



10. Robustness Properties of LQR Regulators

- Nyquist plot for LQR design avoids a circle of unity radius centered at the -1 point as shown in Fig. 7.23.
- This leads to extraordinary phase and gain margin properties.
- Consider the return difference equation defined as the ratio between $i(t) - r(t)$ and $i(t)$

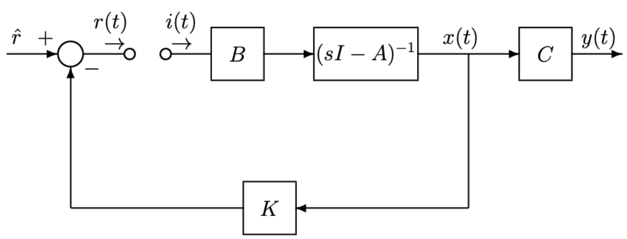


Figure 3.3: Breaking the closed loop LQR system in the input's side

$$r(t) = -K(sI - A)^{-1}Bi(t) \quad \rightarrow \quad \frac{i(t) - r(t)}{i(t)} = 1 + K(sI - A)^{-1}B$$

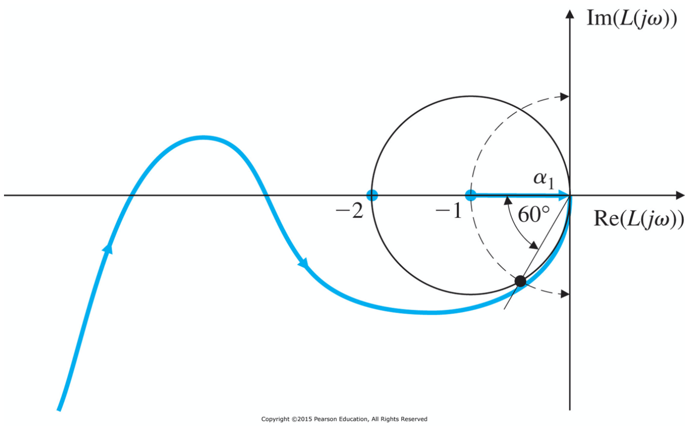
- The magnitude of return difference equation must satisfy

$$|1 + K(j\omega I - A)^{-1}B| \geq 1$$
$$(Re(L(j\omega)) + 1)^2 + (Im(L(j\omega)))^2 \geq 1$$

where

$$L(j\omega) = K(j\omega I - A)^{-1}B$$

– See Fig. 7.89. In other words,



$$\frac{1}{2} < GM < \infty \qquad PM > 60^\circ$$

– These margins are remarkable, and it is not realistic to assume that they can be achieved in practice, because of the presence of modeling errors and lack of sensors.