- (7.5.2) Introducing the Reference Input ($r \neq 0$) with Full-State Feedback ($u = -Kx$)

  1. In order to introduce the reference input into the control law, we use

  $$u = -Kx \qquad \rightarrow \qquad \therefore \quad u = -Kx + \bar{N}r$$

     If the desired final values of the state and the control input are $x_{ss}$ and $u_{ss}$, respectively, then the new control formula should be

  $$u - u_{ss} = -K(x - x_{ss}) \qquad \rightarrow \qquad \therefore \quad u = -Kx + \bar{N}r$$

     so that $u = u_{ss}$, when $x = x_{ss}$ (no error).

  2. To pick the correct final values, we must solve the equations so that the system will have zero steady-state error to any constant input. Namely, $y_{ss} = r_{ss}$, furthermore, we make $x_{ss} = N_x r_{ss}$ and $u_{ss} = N_u r_{ss}$

  $$\begin{aligned} \dot{x} &= Ax + Bu & \qquad 0 &= Ax_{ss} + Bu_{ss} & \qquad 0 &= AN_x r_{ss} + BN_u r_{ss} \\ y &= Cx + Du & \qquad y_{ss} &= Cx_{ss} + Du_{ss} & \qquad r_{ss} &= CN_x r_{ss} + DN_u r_{ss} \end{aligned}$$

     Now we have following matrix equation:

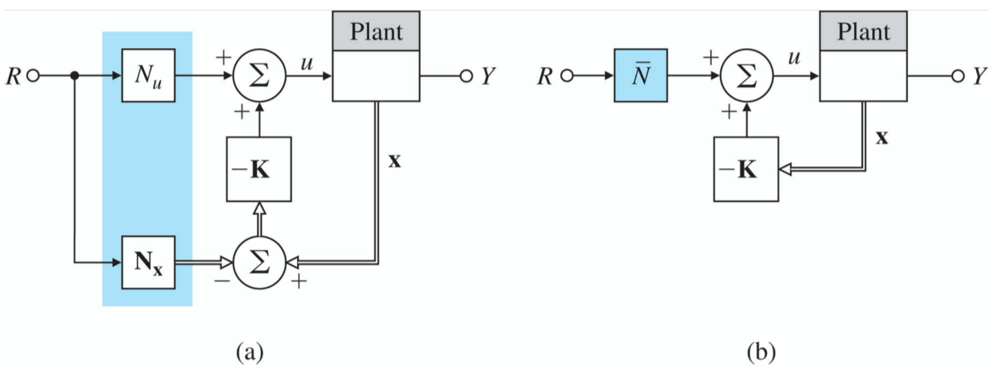
  $$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \rightarrow \qquad \begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

     for $x_{ss} = N_x r_{ss}$ and $u_{ss} = N_u r_{ss}$

3. With $N_x$ and $N_r$, we finally have the basis for introducing the reference input so as to get zero steady-state error to a step input $r_{ss} = r$

$$
\begin{aligned}
u &= u_{ss} - K(x - x_{ss}) \\
&= N_u r - K(x - N_x r) \\
&= -Kx + (N_u + KN_x)r \\
&= -Kx + \bar{N} r
\end{aligned}
$$

where $\bar{N} = N_u + KN_x$. See Fig. 7.15



4. (Example 7.17) Using the results of (Example 7.14), compute the necessary gains for zero steady-state error to a step command at $x_1$ and plot the resulting unit step response? ($\omega_o = 1$)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

The result of (Example 7.14) was $K = [3\omega_o^2\ , 4\omega_o] = [3\ ,\ 4]$. Also we have

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $\bar{N}$ is obtained

$$\bar{N} = N_u + KN_x = 1 + \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 4$$

The set-point regulation controller to step command ($r = 1$) becomes

$$\begin{aligned} u &= -Kx + \bar{N}r = -3x_1 - 4x_2 + 4r \\ &= r + 3(r - y) + 4(\dot{r} - \dot{y}) \end{aligned}$$

because $\dot{r} = 0$, $y = x_1$ and $\dot{y} = \dot{x}_1 = x_2$.

5. (Example 7.18) Compute the input gains necessary to introduce a reference input with zero steady-state error to a step for the DC motor of (Example 5.1). Assume that the state feedback gain is $K = [K_1\ , K_2]$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

$N_x$ and $N_u$ are obtained as

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, $\bar{N}$ is obtained

$$\bar{N} = N_u + KN_x = 0 + \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = K_1$$

The expression for the control using $N_x$ and $N_u$ reduces to

$$u = -Kx + \bar{N}r = -K_1x_1 - K_2x_2 + K_1r = K_1(r - y) + K_2(\dot{r} - \dot{y})$$

where $y = x_1$, $\dot{r} = 0$, and $\dot{y} = \dot{x}_1 = x_2$

6. The closed-loop control system from the reference input $r$ to the output $y$ is

$$\dot{x} = Ax + B(-Kx + \bar{N}r) = (A - BK)x + B\bar{N}r$$
$$y = Cx$$

then the closed-loop poles and zeros become

$$\det[pI - A + BK] = 0 \qquad \det\begin{bmatrix} zI - A + BK & -\bar{N}B \\ C & 0 \end{bmatrix} = 0$$
$$\rightarrow \quad \det\begin{bmatrix} zI - A & -B \\ C & 0 \end{bmatrix} = 0$$

where $p$ and $z$ denote closed-loop poles and zeros, respectively. In fact, the zeros are not changed by the feedback. You can easily check it by $\bar{N}$ column scaling and $BK$ column addition.

# 6 Selection of Pole Locations for Good Design

- Pole placement aims
  - to fix only the undesirable aspects of the open-loop response
  - to avoid large increases in bandwidth
  - to need smaller control effort.

- (7.6.1) Dominant Second-Order Poles

  1. We can pick the low-frequency modes to achieve desired values of $\omega_n$ and $\zeta$ and select the rest of the poles to increase the damping of the high-frequency mode, while holding their frequency constant in order to minimize control effort.
  2. (Example 7.19) Design the feedback control for the drone system (Example 5.12) to have overshoot less than 5% and a rise time less than 1[s]

$$G(s) = \frac{1}{s^2(s+2)}$$

(Solution) We need to memorize the results of chapter 3 as follows: For a 2nd-order system with no finite zeros, the transient response parameters are approximated as follows:

$$\text{rise time} \quad t_r \approx \frac{1.8}{\omega_n} \qquad \text{overshoot} \quad M_p \approx \begin{cases} 5\% & \zeta = 0.7 \\ 10\% & \zeta = 0.6 \\ 16\% & \zeta = 0.5 \\ 35\% & \zeta = 0.3 \end{cases} \qquad \text{settling time} \quad t_s \approx \frac{4.6}{\zeta\omega_n}$$

From the above, we can get

$$\omega_n > \frac{1.8}{1} \quad \rightarrow \quad \omega_n = 2\sqrt{2} \qquad\qquad \zeta \approx 0.7 \quad \rightarrow \quad \theta = 90^\circ - \sin^{-1}\zeta = 45^\circ$$

Thus, we have the desired dominant poles as

$$p_{1,2} = \omega_n(-\cos\theta \pm j\sin\theta) = -2\sqrt{2}\left(\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}\right) = -2 \pm 2j$$

the remaining third pole is chosen so as to be placed far to the left of the dominant pol pair.

$$p_3 = -12$$

Thus, the desired characteristic equation becomes:

$$\alpha_c(s) = (s^2 + 4s + 8)(s + 12) = s^3 + 16s^2 + 56s + 96$$

Now let us obtain the CCF of the system

$$\dot{x} = Ax + Bu = (A - BK)x \qquad\qquad y = Cx$$

where

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$$

The pole placement or Ackerman's formula can be used

$$\det(sI - A + BK) = \det \begin{bmatrix} s + 2 + K_1 & K_2 & K_3 \\ -1 & s & 0 \\ 0 & -1 & s \end{bmatrix} = s^3 + (2 + K_1)s^2 + K_2 s + K_3 = \alpha_c(s)$$

and we have $K_1 = 14$, $K_2 = 56$ and $K_3 = 96$.