제 3 장

Dynamic Response

- There are three domains within which to study dynamic response:
  1. Laplace Transform (s-plane) (LT) (3,4,5장)
  2. Frequency Response (6장)
  3. State Space (7장)
- In this chapter 3,
  1. LT : mathematical tool for transforming differential equations (DE) into an easier-to-manipulate algebraic form
  2. Block diagram manipulation
  3. Transfer Function (TF)
  4. Its simple frequency response
  5. When feedback is introduced, the possibility that system may become unstable is introduced.
  6. Definition of stability and Routh's test
  7. Signal-flow graph (Mason's formula)

1 Review of Laplace Transforms

Two attributes of linear time-invariant (LTI) systems

- A linear system response obeys the principle of superposition
- The response of an LTI system can be expressed as the convolution of input with the unit impulse response of the system

1. Response by Convolution :

- The principle of superposition states that if the system has an input that can be expressed as a sum of signals, then the response of the system can be expressed as the sum of the individual responses to the respective signals.

$$
\begin{aligned}
u_1(t) &\quad\rightarrow\quad y_1(t) \\
u_2(t) &\quad\rightarrow\quad y_2(t) \\
u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t) &\quad\rightarrow\quad y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)
\end{aligned}
$$

Superposition will apply if and only if the system is linear.

(Example 3.1, Superposition) Show that superposition holds for the system modeled by DE

$$\dot{y} + ky = u$$

$$
\begin{aligned}
u_1(t) &\quad\rightarrow\quad y_1 \quad \text{satisfying} \quad \dot{y}_1 + ky_1 = u_1 \\
u_2(t) &\quad\rightarrow\quad y_2 \quad \text{satisfying} \quad \dot{y}_2 + ky_2 = u_2 \\
\alpha_1 u_1(t) + \alpha_2 u_2(t) = u(t) &\quad\rightarrow\quad y = \alpha_1 y_1 + \alpha_2 y_2 \quad \text{satisfying} \quad \dot{y} + ky = u \\
&\quad (\alpha_1 \dot{y}_1 + \alpha_2 \dot{y}_2) + k(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 u_1 + \alpha_2 u_2
\end{aligned}
$$

If we consider new input u as $\alpha_1 u_1(t) + \alpha_2 u_2(t)$, new output y is obtained as linear combinations of individual outputs $\alpha_1 y_1(t) + \alpha_2 y_2(t)$. Thus the superposition holds.

- If the input is delayed or shifted in time, then the output is unchanged except also being shifted by exactly the same amount.

$$\begin{aligned} u_1(t) &\quad\rightarrow\quad y_1(t) \\ u_2(t) = u_1(t-\tau) &\quad\rightarrow\quad y_2(t) = y_1(t-\tau) \end{aligned}$$

(Example 3.2, Time Invariance) Consider when $k(t)$ is dependent on time in Example 3.1

$$\begin{aligned} u_1(t) &\quad\rightarrow\quad \dot{y}_1(t) + k(t)y_1(t) = u_1(t) \\ u_2(t) = u_1(t-\tau) &\quad\rightarrow\quad \dot{y}_2(t) + k(t)y_2(t) = u_2(t) = u_1(t-\tau) \end{aligned}$$

Assume that $y_2(t) = y_1(t-\tau)$, then

$$\dot{y}_1(t-\tau) + k(t)y_1(t-\tau) = u_1(t-\tau)$$

Let us make the change of variable $t-\tau = \eta$, then

$$\dot{y}_1(\eta) + k(\eta+\tau)y_1(\eta) = u_1(\eta)$$

Since $k(\eta+\tau) \neq k(\eta)$, the system is not time-invariant, except when $k(t) = k$ is constant.

- The most common candidates for elementary signals are the impulse and the exponential.
- A short pulse $\delta_\Delta(t)$ is defined as a rectangular pulse having unit area such that

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 \leq t \leq \Delta \\ 0 & \text{otherwise} \end{cases}$$

In addition, as $\Delta \to 0$, the unit impulse is obtained as

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

where it has the property that

$$\begin{cases} \int_{-\infty}^{\infty} \delta(t) dt = 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

- Consider a short pulse approximation $\hat{x}(t)$ to CT (continuous-time) signal $x(t)$,

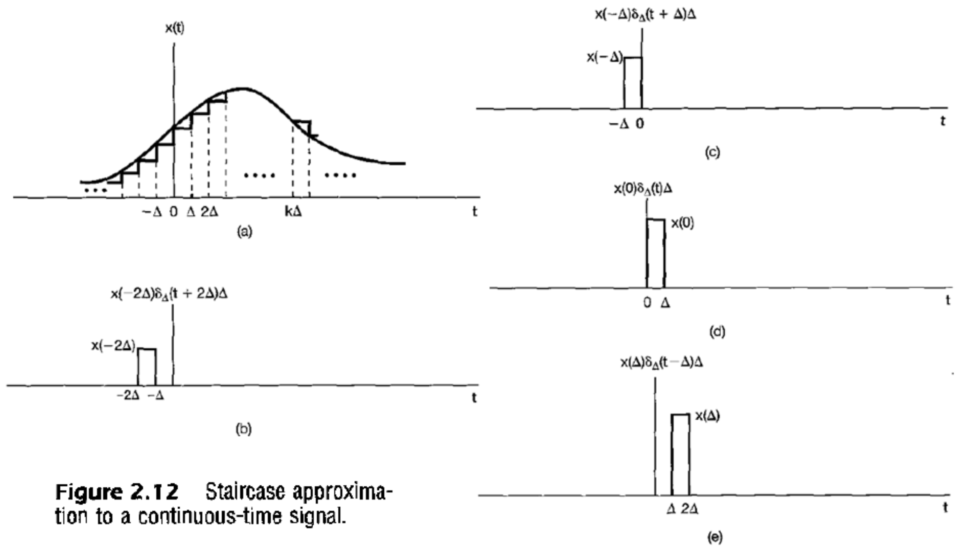

Figure 2.12 Staircase approximation to a continuous-time signal.

Since $\delta_\Delta(t)\ \Delta$ has the "unit amplitude", we have:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta.$$

As $\Delta \to 0$, the approximation $\hat{x}(t)$ recovers the CT signal $x(t)$ as follows:

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau.$$

where it is noted that $\Delta \to d\tau$, $\delta_\Delta(t) \to \delta(t)$, $k\Delta \to \tau$, and $\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} \to \int_{\tau=-\infty}^{\tau=\infty}$.

- This is referred to as *sifting property*. Specifically, the signal $\delta(t-\tau)$ is a unit impulse located at $\tau = t$. Thus, the signal $x(\tau)\delta(t-\tau)$ equals $x(t)\delta(t-\tau)$. Consequently, the integral of this signal from $\tau = -\infty$ to $\tau = \infty$ equals $x(t)$; that is

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau = x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = x(t)$$

- Consider system with input $u(t)$ and output $y(t)$, similarly, with input $\delta(t)$ and output $h(t)$,

$$\begin{array}{ccc} u(t) & \rightarrow & y(t) \\ \delta(t) & \rightarrow & h(t) \end{array} \quad \text{: it is referred to as unit impulse response}$$

From the property of time invariance (TI):

$$\delta(t-\tau) \quad \rightarrow \quad h(t-\tau)$$

From the principle of superposition, we can conclude that

$$\begin{array}{ccc} \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau & \rightarrow & \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau \\ \| & & \| \\ u(t) & \rightarrow & y(t) \end{array}$$

- For an LTI system, the output becomes

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

or by changing of variable as $\tau_1 = t - \tau$,

$$y(t) = \int_{\infty}^{-\infty} u(t-\tau_1)h(\tau_1)(-d\tau_1) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

This is referred to as the *convolution integral*.

(Example 3.3, Convolution) Determine the impulse response for the system

$$\dot{y} + ky = u = \delta(t) \qquad \text{with an initial condition of } y(0-) = 0 \text{ before the impulse}$$

Let us take integral both sides as follows:

$$\int_{0-}^{0+} \dot{y}dt + k\int_{0-}^{0+} ydt = \int_{0-}^{0+} \delta(t)dt$$
$$[y(0+) - y(0-)] + k \times 0 = 1$$
$$\therefore \qquad y(0+) = 1$$

For the positive time, we have the following differential equation:

$$\dot{y} + ky = 0, \qquad y(0+) = 1$$

If we assume a solution $y = Ae^{st}$, then $\dot{y} = Ase^{st}$. The preceding equation becomes

$$Ase^{st} + Ake^{st} = 0 \quad \rightarrow \quad A(s+k)e^{st} = 0 \quad \rightarrow \quad s = -k$$

From the initial condition of $y(0+) = 1$, we can determine $A = 1$ and ultimately we have the unit impulse response of the system as follow:

$$\therefore \qquad h(t) = e^{-kt} \qquad \text{fot} \quad t > 0$$

where $h(t) = 0$ for $t < 0$.

- Let us define the unit-step function

$$1(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

- Solution of (Example 3.3) can be written as one equation using the unit-step function

$$h(t) = \begin{cases} 0 & t < 0 \\ e^{-kt} & t \geq 0 \end{cases} \qquad \rightarrow \qquad h(t) = e^{-kt}1(t)$$

Consider the folllowing convolution:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} e^{-k\tau}1(\tau)u(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-k\tau}u(t-\tau)d\tau = \int_{0}^{\infty} u(t-\tau)h(\tau) = \int_{0}^{\infty} u(\tau)h(t-\tau)d\tau \end{aligned}$$

- If $h(t)$ has the value for negative time, it means that the system response starts before the input is applied. Systems which do this are called non-causal because they do not obey the usual law of cause and effect.
- All physical systems are causal. In most cases of interest we take $t = 0$ as the time when the input starts. In this case, with causal systems, the integral may be written as:

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau \qquad \rightarrow \qquad y(t) = \int_{0}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} u(t-\tau)h(\tau)d\tau$$