제 5 장

Velocity Kinematics and Statics

- This chapter deals with problem of calculating the twist of the end-effector of an open chain from a given set of joint positions and velocities.
- Consider the FK (forward kinematics) with a minimal set of coordinates $x \in \Re^m$ and a set of joint variables $\theta \in \Re^n$

$$x = f(\theta) \qquad \rightarrow \qquad \dot{x} = \frac{\partial f(\theta)}{\partial \theta^T}\frac{\partial \theta}{\partial t} = \frac{\partial f(\theta)}{\partial \theta^T}\dot{\theta} = J(\theta)\dot{\theta}$$

  where $J(\theta) \in \Re^{m \times n}$ is called the Jacobian.
- The Jacobian matrix represents the linear sensitivity of the end-effector velocity $\dot{x}$ to the joint velocity $\dot{\theta}$, and it is a function of the joint variables θ.

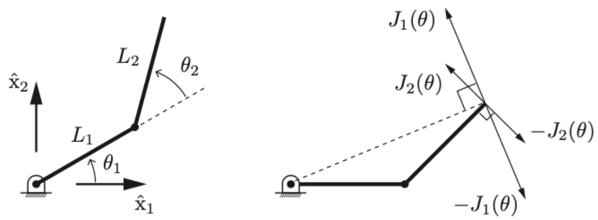


Figure 5.1: (Left) A 2R robot arm. (Right) Columns 1 and 2 of the Jacobian correspond to the endpoint velocity when $\dot{\theta}_1 = 1$ (and $\dot{\theta}_2 = 0$) and when $\dot{\theta}_2 = 1$ (and $\dot{\theta}_1 = 0$), respectively.

- Consider the FK of 2R planar open chain

$$x_1 = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \qquad x_2 = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

$$\dot{x}_1 = -L_1 \sin\theta_1 \dot{\theta}_1 - L_2 \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \qquad \dot{x}_2 = L_1 \sin\theta_1 \dot{\theta}_1 + L_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$

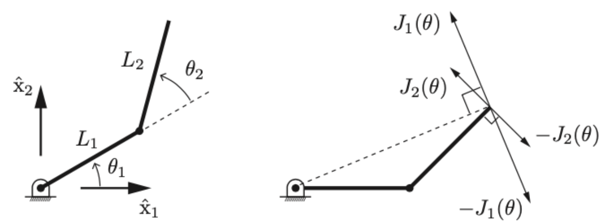
- Velocity kinematics using the Jacobian and its two column vectors

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$v_{tip} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$$

- As long as $J_1(\theta)$ and $J_2(\theta)$ are not collinear, it is possible to generate a tip velocity v_{tip} in any arbitrary direction in the x_1-x_2 plane by choosing appropriate joint velocities $\dot{\theta}_1$ and $\dot{\theta}_2$.
- If θ_2 is $0°$ or $180°$, $J_1(\theta)$ and $J_2(\theta)$ are collinear regardless of θ_1 (singularity). If $\theta_2 = 0$,

$$J_1(\theta) = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1) \end{bmatrix} = \begin{bmatrix} -(L_1 + L_2) \sin\theta_1 \\ (L_1 + L_2) \cos\theta_1 \end{bmatrix} \qquad J_2(\theta) = \begin{bmatrix} -L_2 \sin(\theta_1) \\ L_2 \cos(\theta_1) \end{bmatrix} = \frac{L_2}{L_1 + L_2} J_1(\theta)$$



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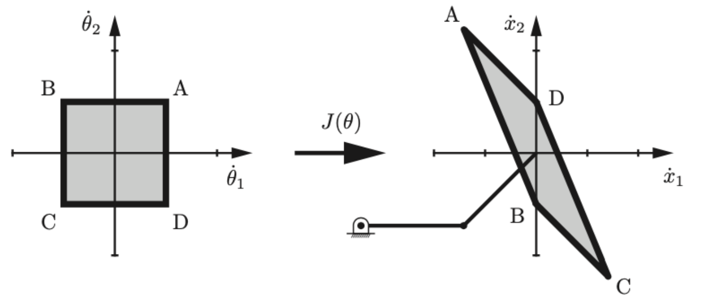


Figure 5.2: Mapping the set of possible joint velocities, represented as a square in the $\dot{\theta}_1$–$\dot{\theta}_2$ space, through the Jacobian to find the parallelogram of possible end-effector velocities. The extreme points A, B, C, and D in the joint velocity space map to the extreme points A, B, C, and D in the end-effector velocity space.

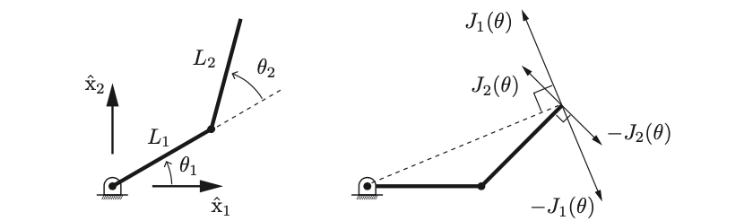
- **With $L_1 = L_2 = 1$, consider the robot at nonsingular configuration $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{4}$**

$$J\left(\begin{bmatrix} 0 \\ \frac{\pi}{4} \end{bmatrix}\right) = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \qquad \rightarrow \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

- **The Jacobian can be used to map bounds on the rotational speed of the joints to bounds on v_{tip}. For example, if the maximal speeds of motors are bounded as $\theta_{max,1} = 10[rad/s]$ and $\theta_{max,2} = 10[rad/s]$**

$$\text{Point A} \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} -14.2 \\ 24.2 \end{bmatrix}$$

$$\text{Point B} \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \begin{bmatrix} -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \end{bmatrix}$$



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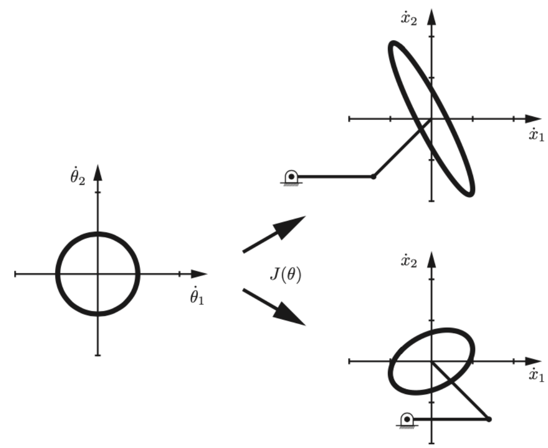


Figure 5.3: Manipulability ellipsoids for two different postures of the 2R planar open chain.

- Rather than mapping a polygon of joint velocities through the Jacobian, we could instead map a unit circle ($\dot{\theta}_1^2 + \dot{\theta}_2^2 = 1$, namely, $\dot{\theta}_1 = \cos\alpha$ and $\dot{\theta}_2 = \sin\alpha$ with $\alpha \in [0, 2\pi)$) of joint velocities in the $\dot{\theta}_1 - \dot{\theta}_2$ plane.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} -0.71(\cos\alpha + \sin\alpha) \\ \cos\alpha + 0.71(\cos\alpha + \sin\alpha) \end{bmatrix}$$

$$\text{if } \alpha = 0 \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.71 \\ 1.71 \end{bmatrix} \qquad \text{if } \alpha = 45^\circ \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1.71 \end{bmatrix} \qquad \text{if } \alpha = 90^\circ \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.71 \\ 0.71 \end{bmatrix}$$

- This circle maps through the Jacobian to an ellipse in the space of tip velocities, and this ellipse is referred to as the manipulability ellipsoid.
- As the manipulator configuration approaches a singularity, the ellipse collapses to a line segment, since the ability of the tip to move in one direction is lost.
- The closer the ellipsoid is to a circle, i.e., the closer the ratio $\frac{l_{max}}{l_{min}}$ is to 1, the more easily can the tip move in arbitrary directions and thus the more removed it is from a singularity.

The Jacobian also plays a central role in static analysis.

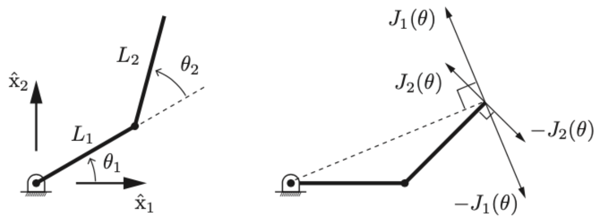
- Suppose that an external force is applied to the robot tip. What are the joint torques required to resist this external force?
- The conservation of power, assuming that negligible power is used to move the robot

$$f_{tip}^T v_{tip} = \tau^T \dot{\theta} \qquad \rightarrow \qquad f_{tip}^T J(\theta)\dot{\theta} = \tau^T \dot{\theta} \qquad \rightarrow \qquad \tau = J^T(\theta) f_{tip}$$

  where the joint torque τ needed to create the tip force f_{tip} is calculated from the equation above.

- If the inverse of $J^T(\theta)$ exists,

$$f_{tip} = J^{-T}(\theta)\tau$$



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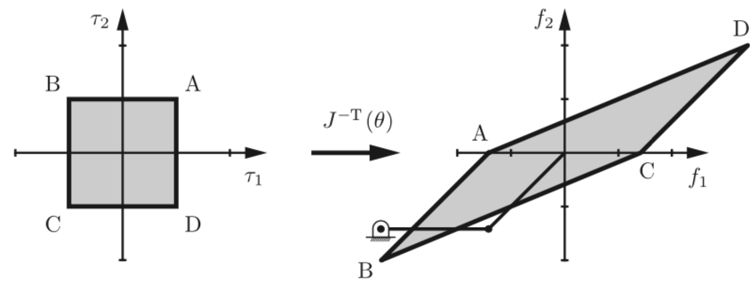


Figure 5.4: Mapping joint torque bounds to tip force bounds.

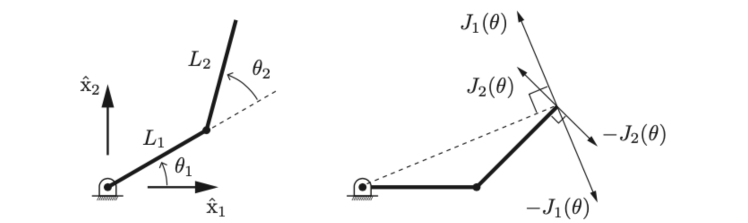
- From the previous example,

$$J\left(\begin{bmatrix} 0 \\ \frac{\pi}{4} \end{bmatrix}\right) = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \rightarrow \tau = J^T(\theta) f_{tip} \rightarrow \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -0.71 & 1.71 \\ -0.71 & 0.71 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \rightarrow \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & -2.41 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

- The inverse of Jacobian transpose can be used to map bounds on motor torques τ to bounds on f_{tip}. For example, if the maximal torques of motors are bounded as $\tau_{max,1} = 10[Nm]$ and $\tau_{max,2} = 10[Nm]$

Point A $$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & -2.41 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} -14.1 \\ 0 \end{bmatrix}$$

Point B $$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & -2.41 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -10 \\ 10 \end{bmatrix} = \begin{bmatrix} -34.1 \\ -20 \end{bmatrix}$$



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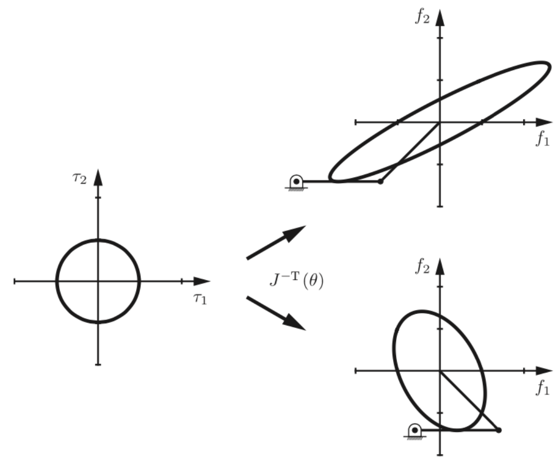

Figure 5.5: Force ellipsoids for two different postures of the 2R planar open chain.

- As for the manipulability ellipsoid, a force ellipsoid can be drawn by mapping a unit circle contour in the $\tau_1 - \tau_2$ plane to an ellipsoid in the $f_1 - f_2$ tip-force plane via the Jacobian transpose inverse $J^{-T}(\theta)$.

- Let us obtain unit circle contour ($\tau_1^2 + \tau_2^2 = 1$, namely, $\tau_1 = \cos\alpha$ and $\tau_2 = \sin\alpha$ with $\alpha \in [0, 2\pi)$) of joint torques in the $f_1 - f_2$ plane.

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & -2.41 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha - 2.41\sin\alpha \\ \cos\alpha - \sin\alpha \end{bmatrix}$$

$$\text{if } \alpha = 0 \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{if } \alpha = 45^\circ \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \text{if } \alpha = 90^\circ \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -2.41 \\ -1 \end{bmatrix}$$

- The force ellipsoid illustrates how easily the robot can generate forces in different directions.

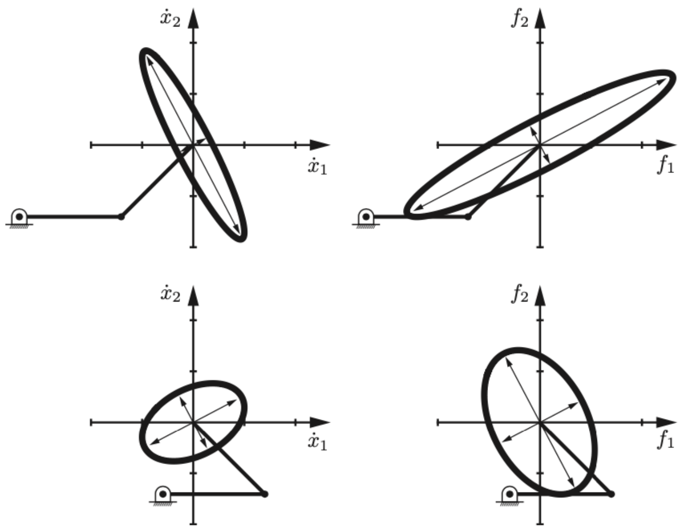

Figure 5.6: Left-hand column: Manipulability ellipsoids at two different arm configurations. Right-hand column: The force ellipsoids for the same two arm configurations.

- As is evident from the manipulability and force ellipsoids, if it is easy to generate a tip velocity in a given direction then it is difficult to generate a force in that same direction, and vice versa.

- For a given robot configuration, the principal axes of the manipulability ellipsoid and force ellipsoid are aligned, and the lengths of the principal semi-axes of the force ellipsoid are the reciprocals of the lengths of the principal semi-axes of the manipulability ellipsoid.

- At a singularity, the manipulability ellipsoid collapses to a line segment. The force ellipsoid, on the other hand, becomes infinitely long in a direction orthogonal to the manipulability ellipsoid line segment (i.e., the direction of the aligned links) and skinny in the orthogonal direction.

- Consider carrying a heavy suitcase with your arm. It is much easier if your arm hangs straight down under gravity (with your elbow fully straightened at a singularity), because the force you must support passes directly through your joints, therefore requiring no torques about the joints.