(NC) 3 Stability of Equilibrium Points / 3.1 Basic Concepts

1. (Stability) is defined at the specific points, not for the system itself.
   cf) the system is stable (X) $\Rightarrow$ the system has any stable equilibrium points (O)

   - An equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise it is unstable
   - It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity

2. (Equilibrium Point) Let us denote an equilibrium point of $\dot{x} = f(x)$ as $\bar{x} \in D$; namely $f(\bar{x}) = 0$.

- The equilibrium point can be always shifted to the origin via a change of variables. For example, when $\bar{x} \neq 0$, if the change of variables $y = x - \bar{x}$ is utilized, since the derivative of y is given by

$$\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) \triangleq g(y) \quad \text{where} \;\; g(0) = 0,$$

  then the system has an equilibrium point at the origin in the new variable y.

- Without loss of generality, we assume that $f(x)$ satisfies $f(0) = 0$ and thus we can check the stability of the origin $x = 0$.

3. (Definition 3.1) ($\epsilon - \delta$ Requirement for Stability) Let f be a locally Lipschitz function defined over a domain $D \in \Re^n$, which contains the origin, and $f(0) = 0$. The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

   - stable, for each $\epsilon > 0$, if $\exists\, \delta > 0$ (dependent on ϵ) $\ni$

   $$\|x(0)\| < \delta(\epsilon) \quad \rightarrow \quad \|x(t)\| < \epsilon, \quad \forall\, t \geq 0$$

   - unstable if it is not stable.
   - asymptotically stable if it is stable and δ can be chosen $\ni$

   $$\|x(0)\| < \delta \quad \rightarrow \quad \lim_{t \to \infty} x(t) = 0$$

   a) For any ϵ, we must produce δ (dependent on ϵ) such that a trajectory starting in a δ neighborhood of the origin will never leave the ϵ neighborhood.

   b) Trying to apply $\epsilon - \delta$ theory becomes actually finding all solutions of the state equation, but it may be difficult or even impossible.

   c) As an alternative, Lyapunov's method provides us with a tool to investigate stability of equilibrium points w/o solving the state equation.

4. (Scalar System) For given one-dimensional system
   - the $\epsilon - \delta$ requirement for stability is violated if $xf(x) > 0$. (see Figure 3.1)
   - a necessary condition for the origin to be stable is to have $xf(x) \leq 0$ in some neighborhood of the origin. (see Figure 3.2/3.3)
   - the origin will be asymptotically stable if and only if $xf(x) < 0$ in some neighborhood of the origin. (see Figure 3.3)
   - Let us guess any criterion about the stability using $\dot{x} = f(x)$

$$xf(x) = x\dot{x} \triangleq \dot{V}(x)$$
$$\Downarrow$$
$$V(x) = \frac{1}{2}x^2$$

     - if $\dot{V} > 0$, unstable
     - if $\dot{V} \leq 0$, stable
     - if $\dot{V} < 0$, asymptotically stable

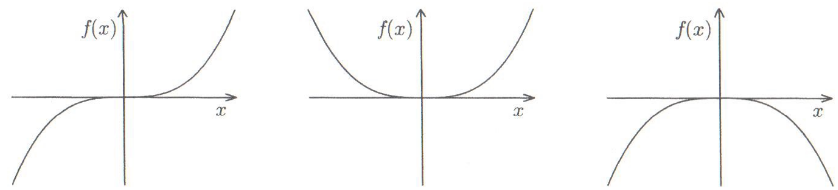


Figure 3.1: Examples of $f(x)$ for which the origin of $\dot{x} = f(x)$ is unstable.

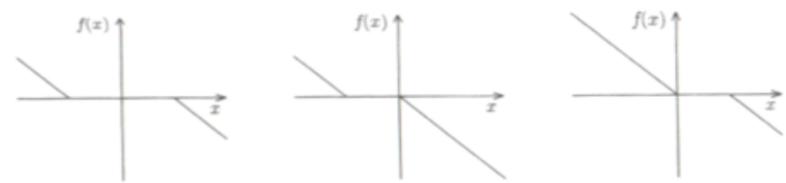


Figure 3.2: Examples of $f(x)$ for which the origin of $\dot{x} = f(x)$ is stable but not asymptotically stable.

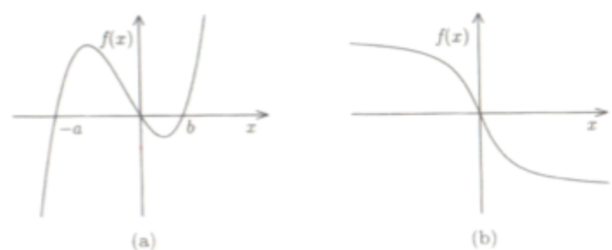


Figure 3.3: Examples of $f(x)$ for which the origin of $\dot{x} = f(x)$ is asymptotically stable.

5. (Region of Attraction) In Figure 3.3(a), if $x_0 = x(0)$ exists in the set $\{-a < x < b\}$, then $x(t) \to 0$ as $t \to \infty$, otherwise it is unstable.

6. (Definition 3.2) (Region of Attraction) Let the origin be an asymptotically stable equilibrium point of the system $\dot{x} = f(x)$, where f is locally Lipschitz function defined over a domain $D \in \Re^n$ that contains the origin. Then

   - the region of attraction of the origin is the set of all points x_0 in D such that the solution of $\dot{x} = f(x)$ starting at $x(0) = x_0$ converges to the origin as t tends to infinity.
   - the origin is globally asymptotically stable if its region of attraction is the whole space $\Re^n$

7. To begin with, let us deal with how to obtain the solution of linear system.

8. (Linear Systems) For the diagonlization, let us obtain the eigenvalue decomposition of A in either real or complex number domain

$$A = M^{-1}\Lambda M \quad \leftrightarrow \quad \Lambda = MAM^{-1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

   where $\lambda_1, \lambda_2 \in \Re$ or $\lambda_{1,2} = \alpha \pm j\beta \in \mathbb{C}$. For linear time-invariant system, the solution is given

$$\dot{x} = Ax \quad \rightarrow \quad x(t) = e^{At}x(0) = M^{-1}e^{\Lambda t}Mx(0) \tag{24}$$

9. (Theorem 3.1) The equilibrium point $x = 0$ of $\dot{x} = Ax$ is stable if and only if

   - all eigenvalues of $A \in \Re^{n\times n}$ satisfy $Re[\lambda_i] \leq 0$ and
   - for every eigenvalue with $Re[\lambda_i] = 0$ and algebraic multiplicity $q_i \geq 2$, $rank(A-\lambda_i I) = n-q_i$

   The equilibrium point $x = 0$ is globally asymptotically stable if and only if all eigenvalues of A satisfy $Re[\lambda_i] < 0$.

10. (Example 3.1) Assume two same systems having following form are connected by series or by parallel. Check the stability of series- or parallel-connected system?

$$\dot{x}_1 = Ax_1 + Bu_1 \qquad \dot{x}_2 = Ax_2 + Bu_2$$
$$y_1 = Cx_1 \qquad y_2 = Cx_2$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Since the series-connected ($u_2 = y_1$) or parallel-connected ($u_2 = u_1$) system has

$$\dot{x}_1 = Ax_1 + Bu_1 \qquad \dot{x}_1 = Ax_1 + Bu_1$$
$$\dot{x}_2 = Ax_2 + Bu_2 = Ax_2 + B(Cx_1) = BCx_1 + Ax_2 \qquad \dot{x}_2 = Ax_2 + Bu_2$$

the system matrix will be either

$$A_s = \begin{bmatrix} A & 0 \\ BC & A \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \qquad A_p = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

- The matrices A_p and A_s have the same eigenvalues on $\pm j$ with multiplicity $q_i = 2$, for $i = 1, 2$. For $\lambda_1 = j$,

$$rank(A_s - jI) = 3 \neq 2 = n - q_1 \qquad rank(A_p - jI) = 2 = 2 = n - q_1$$

- By Theorem 3.1, the origin of series-connected system is unstable, but the origin of parallel-connected system is stable

11. (Hurwitz) When $Re[\lambda_i] < 0$ for $i = 1, \cdots, n$, A is called a Hurwitz matrix. The origin of $\dot{x} = Ax$ is asymptotically stable if and only if A is Hurwitz. In this case, its solution satisfies the inequality

$$\|x(t)\| \leq k\|x(0)\|e^{\lambda t}, \quad \forall\ t \geq 0 \quad \text{with } k > 0 \text{ and } \lambda < 0 \tag{25}$$

12. (Definition 3.3) (Exponential Stability) Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset \Re^n$, which contains the origin, and $f(0) = 0$. The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

   - exponentially stable if $\exists$ positive c, k and $\lambda \ni$ inequality Eq. (25) is satisfied $\forall\ \|x(0)\| < c$.
   - globally exponentially stable if the inequality is satisfied for every initial state $x(0)$.

13. (Example 3.2) Show that the origin of $\dot{x} = -x^3$ is asymptotically stable, but not exponentially stable ?

   (1) $f(x) = -x^3$ is locally Lipschitz b/c $f'(x) = -3x^2$ is continuous and locally bounded for all x in a domain $D \subset \Re$.

   (2) Since $xf(x) < 0$, the origin is asymptotically stable. For given initial condition $x(0) = a$, the solution cannot leave the compact set $\{|x| \leq |a|\}$. Thus we conclude by Lemma 1.3 that it has a unique solution for all $t \geq 0$

$$\frac{dx}{dt} = -x^3 \;\rightarrow\; -\frac{dx}{x^3} = dt \;\rightarrow\; -\int_{x(0)}^{x(t)} x^{-3}dx = \int_0^t dt \;\rightarrow\; \left.\frac{1}{2x^2}\right|_{x(0)}^{x(t)} = t \;\rightarrow\; x(t) = \frac{x(0)}{\sqrt{1 + 2tx^2(0)}}$$

   (3) Since the solution does not satisfy inequality of the form (25), we know that it is asymptotically stable, not exponentially stable.

(NC) 3.2 Linearization

1. Stability can be easily checked by seeing the local behavior (convergence or divergence) near the specific point.

2. (Theorem 3.2) (Lyapunov's Indirect Theorem) Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x} = f(x)$, where f is continuously differentiable in a neighborhood of the origin.

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0}$$

   - the origin is exponentially stable if and only if $Re[\lambda_i] < 0$ for all eigenvalues of A
   - the origin is unstable if $Re[\lambda_i] > 0$ for one or more of the eigenvalues.
   - but, theorem does not say anything about the case when $Re[\lambda_i] \leq 0$ for all i. In this case, linearization fails to determine the stability.

3. (Example 3.4) The pendulum system has two equilibrium points at $(0, 0)$ and $(\pi, 0)$, with $b > 0$.

$$\dot{x}_1 = x_2 \qquad\qquad \dot{x}_2 = -\sin x_1 - bx_2$$

   - Jacobian matrix is $A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -b \end{bmatrix}$
   - At $(0, 0)$, $A = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix}$ and $\lambda_{1,2} = -0.5b \pm 0.5\sqrt{b^2 - 4}$. Since all eigenvalues with $Re[\lambda_i] < 0$, the origin is exponentially stable.
   - At $(\pi, 0)$, $A = \begin{bmatrix} 0 & 1 \\ 1 & -b \end{bmatrix}$ and $\lambda_{1,2} = -0.5b \pm 0.5\sqrt{b^2 + 4}$. Since one eigenvalue with $Re[\lambda_i] > 0$, the $(\pi, 0)$ is unstable.

(NC) 3.3 Lyapunov's Method

1. Reconsider the pendulum equation

$$\dot{x}_1 = x_2 \qquad\qquad \dot{x}_2 = -\sin x_1 - bx_2$$

   we have argued that the origin is stable when $b = 0$ and asymptotically stable when $b > 0$

   - by drawing phase portraits (Example 2.3)
   - by linearization (Example 3.4).
   - As an another approach, the energy concept can be used to determine the stability.

2. (Energy Function) Consider the kinetic energy plus potential energy

$$E(x) = \frac{1}{2}x_2^2 + (1 - \cos x_1) \quad \Leftarrow \quad \frac{1}{2}m(l\dot{\theta})^2 + mg(l - l\cos\theta)$$

   By examining the derivative of E along the trajectories of the system, it is possible to determine the stability of the equilibrium point.

   - When $b = 0$, the origin $x = 0$ is a stable equilibrium point b/c

$$\frac{dE}{dt} = x_2\dot{x}_2 + \sin x_1 \dot{x}_1 = -x_2 \sin x_1 + \sin x_1 x_2 = 0$$

   - When $b > 0$, it is a stable equilibrium point b/c

$$\frac{dE}{dt} = x_2\dot{x}_2 + \sin x_1 \dot{x}_1 = -x_2 \sin x_1 - bx_2^2 + \sin x_1 x_2 = -bx_2^2 \leq 0$$

     actually, when $b > 0$, the origin is asymptotically stable, although we cannot show it by using energy function. (LaSalle's Theorem)

3. In 1892, Lyapunov showed that certain other functions could be used instead of energy to determine stability of an equilibrium point.

4. (Theorem 3.3) (Lyapunov's Theorem) Let f be a locally Lipschitz function defined over a domain $D \subset \Re^n$, which contains the origin, and $f(0) = 0$.
Let $V(x)$ be a continuously differentiable function defined over $D \ni$

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \forall x \in D \text{ with } x \neq 0 \tag{26}$$

$$\dot{V}(x) \leq 0 \quad \forall x \in D \tag{27}$$

Then the origin is a stable equilibrium point of $\dot{x} = f(x)$.
Moreover, if Eq. (26) holds and

$$\dot{V}(x) < 0 \quad \forall x \in D \text{ with } x \neq 0 \tag{28}$$

then the origin is asymptotically stable
Furthermore, if $D = \Re^n$, Eqs. (26) and (28) hold $\forall x \neq 0$, and

$$\|x\| \to \infty \quad \Rightarrow \quad V(x) \to \infty \tag{29}$$

then the origin is globally asymptotically stable

- $V(0) = 0$ and $V(x) > 0$ ($V(x) < 0$) for $x \neq 0$: is said to be positive (negative) definite
- $V(0) = 0$ and $V(x) \geq 0$ ($V(x) \leq 0$) for $x \neq 0$: is said to be positive (negative) semidefinite
- If $V(x)$ does not have a definite sign, it is said to be indefinite
- $V(x) \to \infty$ as $\|x\| \to \infty$: is said to be radially unbounded.

5. (Rephrasing Lyapunov's Theorem)

   - The origin is stable if $\exists$ a continuously differentiable $V(x) > 0$ with $V(0) = 0 \ni \dot{V} \leq 0$.
   - It is asymptotically stable if $\exists$ continuously differentiable $V(x) > 0$ with $V(0) = 0 \ni \dot{V} < 0$.
   - It is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded

6. (How to Check Positive Definiteness) For the quadratic form

$$V(x) = x^T P x$$

   where P is a real symmetric matrix, $V(x) > 0$ if and only if all the eigenvalues of P are positive, which is also true if and only if all the leading principal minors of P are positive.

7. (Example 3.5) Find the conditions for $V(x) > 0$ and $V(x) < 0$, respectively?

$$V(x) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

   The leading principal minors of P are $a > 0$, $a^2 > 0$, and $a(a^2 - 5) > 0$
   The leading principal minors of $-P$ are $-a > 0$, $a^2 > 0$, and $-a(a^2 - 5) > 0$

   - $a > \sqrt{5}$, for $V(x) > 0$
   - $a < -\sqrt{5}$, for $V(x) < 0$

8. (Advantage and Disadvantage of Lyapunov Theorem) Lyapunov's Theorem can be applied w/o solving the differential equation $\dot{x} = f(x)$, but there is no systematic method for finding Lyapunov functions

9. (Example 3.6) Find the Lyapunov function of pendulum system ($b > 0$):

$$\dot{x}_1 = x_2 \qquad\qquad \dot{x}_2 = -\sin x_1 - bx_2$$

- Starting from the energy $E(x) = \frac{1}{2}x_2^2 + (1 - \cos x_1)$, let us replace the term $\frac{1}{2}x_2^2$ by the more general quadratic form $\frac{1}{2}x^T Px$ for some 2×2 positive definite matrix P:

$$V(x) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (1 - \cos x_1)$$

- For the positive definiteness of $V(x)$: $p_{11} > 0$ and $p_{11}p_{22} - p_{12}^2 > 0$
- For the negative definiteness of $\dot{V}(x)$

$$\begin{aligned}\dot{V}(x) &= (p_{11}x_1 + p_{12}x_2)\dot{x}_1 + (p_{12}x_1 + p_{22}x_2)\dot{x}_2 + \sin x_1 \dot{x}_1 \\ &= [(1 - p_{22})x_2 - p_{12}x_1]\sin x_1 + (p_{11} - bp_{12})x_1x_2 + (p_{12} - bp_{22})x_2^2\end{aligned}$$

- Let us take $p_{22} = 1, p_{11} = \frac{1}{2}b^2, p_{12} = \frac{1}{2}b$, then we have

$$\dot{V} = -\frac{1}{2}bx_1 \sin x_1 - \frac{1}{2}bx_2^2 < 0, \quad \text{since } \; x_1 \sin x_1 > 0 \quad \forall \; |x_1| < \pi$$

- Taking $D = \{|x_1| < \pi\}$, by Theorem 3.3, the origin is asymptotically stable

10. One systematic way to find Lyapunov function is a Variable Gradient Method, although it brings very complex and tedious calculations.

11. (Variable Gradient Method) It is useful in searching for a Lypaunov function. Let $V(x)$ be a scalar function of x and $g(x)^T \triangleq \frac{\partial V}{\partial x}$. Notice that $\frac{\partial V}{\partial x}$ is defined as a row vector. The derivative of $V(x)$ is given by

$$\dot{V}(x) = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}f(x) = g(x)^T f(x)$$

- It is not difficult to verify that $g(x)$ is the gradient of a scalar function if and only if the Jacobian matrix is symmetric; that is

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall i, j = 1, \cdots, n \tag{30}$$

- We start by choosing $g(x) \ni \dot{V}(x) = g(x)^T f(x) < 0$.
- Usually, the function $V(x)$ is chosen as follow

$$V(x) = \int_0^{x_1} g_1(y_1, 0, \cdots, 0)dy_1 + \int_0^{x_2} g_2(x_1, y_2, \cdots, 0)dy_2 + \cdots + \int_0^{x_n} g_n(x_1, x_2, \cdots, x_{n-1}, y_n)dy_n \tag{31}$$

- By leaving some parameters of $g(x)$ undetermined, one would try to choose them to endure that $V(x) > 0$.

12. (Example 3.7) Show that the origin is asymptotically stable using the gradient variable method?

$$\dot{x}_1 = x_2 \qquad\qquad \dot{x}_2 = -h(x_1) - ax_2$$

where $a > 0$, $h(\cdot)$ is locally Lipschitz, $h(0) = 0$, and $yh(y) > 0\ \forall y \neq 0, y \in (-b, c)$.

- To apply the variable gradient method, let us try: $g(x) = \frac{\partial V}{\partial x} = \begin{bmatrix} \phi_1(x_1) + \psi_1(x_2) \\ \phi_2(x_1) + \psi_2(x_2) \end{bmatrix}$
- From the symmetric condition we have

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}, \quad \rightarrow \quad \frac{\partial \psi_1(x_2)}{\partial x_2} = \frac{\partial \phi_2(x_1)}{\partial x_1} \triangleq \gamma, \quad \rightarrow \quad \psi_1(x_2) \triangleq \gamma x_2, \ \text{ and } \ \phi_2(x_1) \triangleq \gamma x_1$$

- By choosing $g(x) \ni \dot{V}(x) = g(x)^T f(x) < 0$, we have

$$\begin{aligned}
\dot{V} &= [\phi_1(x_1) + \psi_1(x_2)]\dot{x}_1 + [\phi_2(x_1) + \psi_2(x_2)]\dot{x}_2 = [\phi_1(x_1) + \gamma x_2]x_2 - [\gamma x_1 + \psi_2(x_2)]h(x_1) - [\gamma x_1 + \psi_2(x_2)]ax_2 \\
&= \gamma x_2^2 - \gamma x_1 h(x_1) - a\gamma x_1 x_2 + \phi_1(x_1)x_2 - [h(x_1) + ax_2]\psi_2(x_2) \ \rightarrow \ \psi_2(x_2) \triangleq \delta x_2, \ \text{and } \phi_1(x_1) \triangleq a\gamma x_1 + \delta h(x_1) \\
&= -\gamma x_1 h(x_1) - (a\delta - \gamma)x_2^2 \quad \text{and} \quad g(x) \triangleq \begin{bmatrix} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}
\end{aligned}$$

- By integrating, we have

$$\begin{aligned}
V(x) &= \int_0^{x_1} g_1(y_1, 0)dy_1 + \int_0^{x_2} g_2(x_1, y_2)dy_2 = \int_0^{x_1} [a\gamma y_1 + \delta h(y_1)]dy_1 + \int_0^{x_2} [\gamma x_1 + \delta y_2]dy_2 \\
&= \frac{a\gamma}{2}x_1^2 + \delta \int_0^{x_1} h(y_1)dy_1 + \gamma x_1 x_2 + \frac{\delta}{2}x_2^2 = \frac{1}{2}x^T \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix} x + \int_0^{x_1} h(y)dy
\end{aligned}$$

- By choosing $\delta > 0$ and $0 < \gamma < a\delta$, $V(x) > 0$ and $\dot{V}(x) < 0$ are ensured over the domain $D = \{-b < x_1 < c\}$. Thus the origin is asymptotically stable.

(NC) 3.4 The Invariance Principle

1. (Revisited Example 3.6)

$$\dot{x}_1 = x_2 \qquad\qquad \dot{x}_2 = -\sin x_1 - bx_2$$

   - The energy Lyapunov function proves just that it is not asymptotically stable but stable

$$E(x) = \frac{1}{2}x_2^2 + (1 - \cos x_1) \quad \rightarrow \quad \frac{dE}{dt} = -bx_2^2 \leq 0$$

   - When $\dot{E} = 0$, it means $x_2 = 0$. From the dynamics, we have

$$x_2 = 0 \quad \Rightarrow \quad \dot{x}_2 = 0 \quad \Rightarrow \quad \sin x_1 = 0 \;\; \text{over } D = \{|x_1| < \pi\}$$

   - The system can maintain the $\dot{V}(x) = 0$ condition only at the origin $x = 0$.

2. (LaSalle's Invariance Principle) If we can find $V(x) > 0 \ni \dot{V}(x) \leq 0$ and if we can establish that no trajectory can stay identically at points where $\dot{V}(x) = 0$ except $x = 0$, then the origin is asymptotically stable.

3. (Positively Invariant Set) Equilibrium points and limit cycles are invariant sets, since any solution starting in the set remains in it for all $t \in \Re$. The set $\Omega_c = \{V(x) \leq c\}$ satisfying $\dot{V}(x) \leq 0$ is positively invariant set since a solution starting in Ω_c remains in Ω_c $\forall t \geq 0$.

4. (LaSalle's Invariance Theorem) Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset \Re^n$, which contains the origin, and $f(0) = 0$. Let $V(x)$ be a continuously differentiable positive definite function over $D \ni \dot{V}(x) \leq 0$ in D. Let $S = \{x \in D | \dot{V} = 0\}$ and suppose that no solution can staty identically in S, other than the trivial solution $x(t) = 0$. Then the origin is an asymptotically stable equilibrium point.
   Finally, if $D = \Re^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable.

5. (Example 3.8)

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

   where h_1 and h_2 are locally Lipschitz and satisfies $h_i(0) = 0$, $yh_i(y) > 0$, for $0 < |y| < a$.

   - Energy Lyapunov function can be taken : $V(x) = \frac{1}{2}x_2^2 + \int_0^{x_1} h_1(y)dy > 0$
   - Let $D = \{|x_1| < a, |x_2| < a\}$; $V(x) > 0$ and

$$\dot{V}(x) = x_2\dot{x}_2 + h_1(x_1)\dot{x}_1 = -x_2h_2(x_2) \leq 0$$

     note that $\dot{V} = 0$ means $x_2 = 0$ since $h_2(x_2) \neq 0$ except $x_2 = 0$
   - Hence $S = \{x \in D | x_2 = 0\}$. Let $x(t)$ be a solution that belongs identically to S

$$x_2 = 0 \quad \Rightarrow \quad \dot{x}_2 = 0 \quad \Rightarrow \quad h_1(x_1) = 0 \quad \Rightarrow \quad x_1 = 0$$

   - Only solution that can stay identically in S is $x(t) = 0$, and thus the origin is asymptotically stable.

(NC) 3.5 Exponential Stability

1. We have seen in (Theorem 3.2) that the origin of $\dot{x} = f(x)$ is exponentially stable if and only if the Jacobain is Hurwitz. This result, however, is local.

2. (Theorem 3.6) Let $f(x)$ be a locally Lipschitz function defined over a domain $D \in \Re^n$, which contains the origin, and $f(0) = 0$. Let $V(x)$ be a continuously differentiable function defined over $D \ni$

$$k_1\|x\|^a \leq V(x) \leq k_2\|x\|^a \tag{32}$$

$$\dot{V}(x) \leq -k_3\|x\|^a \quad \forall x \in D \tag{33}$$

   where k_1, k_2, k_3 and a are positive constants. Then the origin is an exponentially stable equilibrium of $\dot{x} = f(x)$.
   If the assumptions hold globally, the origin will be globally exponentially stable.

3. (Example 3.10)

$$\dot{x}_1 = x_2 \qquad\qquad \dot{x}_2 = -h(x_1) - x_2$$

where h is locally Lipschitz, $h(0) = 0$ and $c_1 y^2 \leq yh(y) \leq c_2 y^2$ with positive constants c_1 and c_2 for all y

- Take the Lyapunov function as following form

$$V(x) = x^T P x + 2\int_0^{x_1} h(y)dy \quad \text{where } P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

  whose derivative satisfies

$$\dot{V}(x) = (x_1 + x_2)\dot{x}_1 + (x_1 + 2x_2)\dot{x}_2 + 2h(x_1)\dot{x}_1 = -x_1 h(x_1) - x_2^2$$

- Then we know

$$\begin{aligned} x^T P x \leq V(x) &\leq x^T P x + c_2 x_1^2 \\ \dot{V}(x) &\leq -c_1 x_1^2 - x_2^2 \end{aligned}$$

- Using the fact that $\lambda_{min}(P)\|x\|^2 \leq x^T P x \leq \lambda_{max}(P)\|x\|^2$, we can rewrite above inequalities as follows:

$$\begin{aligned} \lambda_{min}(P)\|x\|^2 \leq V(x) &\leq (\lambda_{max}(P) + c_2)\|x\|^2 \\ \dot{V}(x) &\leq -c_1 x_1^2 - x_2^2 \end{aligned}$$

- Hence, by Theorem 3.6, the origin is globally exponentially stable.

4. (For Linear Systems) By applying Theorem 3.6 with $V(x) = x^T P x$, where $P = P^T > 0$, the derivative of V along $\dot{x} = Ax$ is given by

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + PA) x \triangleq -x^T Q x$$

where $Q = Q^T > 0$ defined by

$$A^T P + PA = -Q \tag{34}$$

If $\exists\ Q = Q^T > 0$, we can say by Theorem 3.3 or 3.6 that the origin is globally exponentially stable; that is, A is Hurwitz. Equation (34) is called the Lyapunov equation.

5. (Theorem 3.7) A matrix A is Hurwitz if and only if, for every $Q = Q^T > 0$, $\exists\ P = P^T > 0$ that satisfies $A^T P + PA + Q = 0$. Moreover, if A is Hurwitz, then P is unique.

6. If the linearization is applied to the nonlinear system $\dot{x} = f(x)$ and $f(0) = 0$, we have

$$\dot{x} = f(x) = [A + G(x)]x \qquad \text{where } A = \left.\frac{\partial f}{\partial x}\right|_{x=0} \text{ and } G(x) \to 0 \text{ as } x \to 0$$

   - When A is Hurwitz, we can solve Lyapunov equation $A^T P + PA + Q = 0$ for $Q > 0$, and use $V(x) = x^T P x$ as a Lyapunov function candidate for the nonlinear system. Then

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T [A + G]^T P x + x^T P [A + G] x = -x^T Q x + 2 x^T P G x$$

   - Since $G(x) \to 0$ as $x \to 0$, for given any $0 < k < 1$, we can find $r > 0 \ni 2\|PG\| < k\lambda_{min}(Q)$ in domain $D = \{\|x\| < r\}$. The origin is exponentially stable in D (region of attraction) b/c

$$\dot{V}(x) \leq -(1-k)\lambda_{min}(Q)\|x\|^2$$

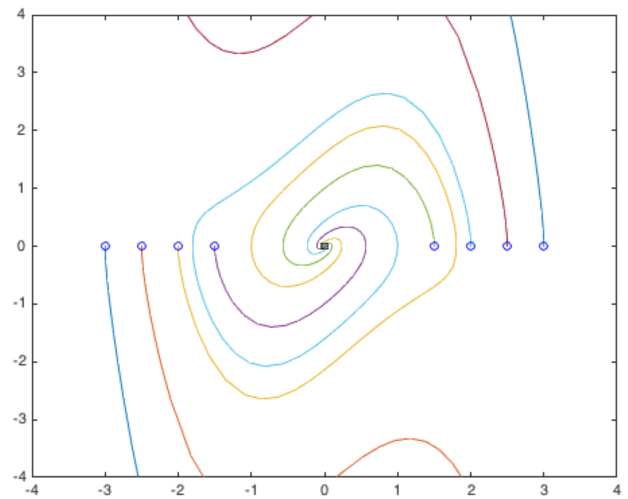
(NC) 3.6 Region of Attraction

1. (Example 3.11) Draw the region of attraction of the following system

$$\dot{x}_1 = -x_2 \qquad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2 \tag{35}$$

- Van der Pol oscillator in reverse time.

```
v = @(t,x) [-x(2); x(1) + (x(1)^2-1)*x(2) ];
for x10 = [-3 -2.5 -2 -1.5 1.5 2 2.5 3]
        [ts,ys] = ode45(v,[0,20],[x10;0]);
        axis([-4 4 -4 4])
        plot(ys(:,1),ys(:,2))
        hold on
        plot(ys(1,1),ys(1,2),'bo') % starting
        hold on
        plot(ys(end,1),ys(end,2),'ks') % ending
end
```



- The origin is stable focus surrounded by an unstable limit cycle.
- The region of attraction is all trajectories in the interior of the limit cycle spiral towards the origin.

2. Lyapunov's method can be used to estimate the region of attraction.

   - The simplest estimate is the set $\Omega_c = \{V(x) < c\}$ satisfying $\dot{V}(x) \leq 0$ with $\Omega_c \subset D$.
   - For a quadratic Lyapunov function $V(x) = x^T P x$ and $D = \{\|x\| < r\}$ satisfying $\dot{V}(x) \leq 0$, we can ensure that $\Omega_c \subset D$ by choosing

$$c < \min_{\|x\|=r} x^T P x = \lambda_{min}(P) r^2 \tag{36}$$

     where $\Omega_c = \{V(x) < c\}$ satisfying $\dot{V}(x) \leq 0$ is the positively invariant set in D
   - For $D = \{|b^T x| < r\}$, where $b \in \Re^n$, since

$$\min_{|b^T x|=r} x^T P x = \frac{r^2}{b^T P^{-1} b} \tag{37}$$

     Ω_c will be a subset of $D = \{|b_i^T x| < r_i, i = 1, \cdots, p\}$, if we choose

$$c < \min_{1 \leq i \leq p} \frac{r_i^2}{b_i^T P^{-1} b_i} \tag{38}$$

   - Whenever $A = \frac{\partial f}{\partial x}|_{x=0}$ is Hurwitz, we can estimate the region of attraction of the origin.

3. (Example 3.14) Find the region of attraction of the following system?

$$\dot{x}_1 = -x_2 \qquad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$

- The origin is asymptotically stable over the domain $D = \{\|x\| \leq r\}$ because the linearized matrix A is Hurwitz: (cf. eig(A) in matlab)

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \qquad \lambda_{1,2}(A) = -0.5 \pm j\frac{\sqrt{3}}{2}$$

- A Lyapunov function can be found by taking $Q = I$ and solving the Lyapunov equation $PA + A^T P = -I$ for P: (cf. lyap(A,eye(2,2)) in matlab)

$$P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

- The derivative of $V(x) = x^T P x$ along the trajectories of the system is given by

$$\begin{aligned} \dot{V}(x) &= 2(1.5x_1 - 0.5x_2)\dot{x}_1 + 2(-0.5x_1 + x_2)\dot{x}_2 \\ &= -2(1.5x_1 - 0.5x_2)x_2 + 2(-0.5x_1 + x_2)(x_1 + (x_1^2 - 1)x_2) \\ &= -(x_1^2 + x_2^2) - x_1^2 x_2(x_1 - 2x_2) \end{aligned}$$

- By using $|x_1| \leq \|x\|$, $|x_1 x_2| \leq 0.5\|x\|^2$ and $|x_1 - 2x_2| \leq \sqrt{5}\|x\|$, we have

$$\begin{aligned} \dot{V}(x) &\leq -\|x\|^2 + |x_1||x_1 x_2||x_1 - 2x_2| \\ &\leq -\|x\|^2 + 0.5\sqrt{5}\|x\|^4 \\ &= -(1 - 0.5\sqrt{5}\|x\|^2)\|x\|^2 \end{aligned}$$

- Now we have the domain $D = \{\|x\| < \sqrt{\frac{2}{\sqrt{5}}} = 0.9457\}$ satisfying $\dot{V}(x) \leq 0$
- Furthermore, the invariant set $\Omega_c = \{V(x) < c\}$ satisfying $\dot{V}(x) \leq 0$ is obtained by choosing

$$c < \min_{\|x\|=r} x^T P x = \lambda_{min}(P) r^2 = 0.691 \times \frac{2}{\sqrt{5}} \approx 0.618 \tag{39}$$

- Thus the set Ω_c with $c = 0.61$ is an estimate of the region of attraction.

4. (Example 3.15) Find the region of attraction of the following system?

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -4(x_1 + x_2) - h(x_1 + x_2)$$

where h is locally Lipschitz function that satisfies

$$h(0) = 0; \qquad uh(u) \geq 0 \quad \forall \ |u| \leq 1$$

- Let us try the quadratic function

$$V(x) = x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = 2x_1^2 + 2x_1x_2 + x_2^2 > 0$$

- Its derivative is obtained as

$$
\begin{aligned}
\dot{V}(x) &= (4x_1 + 2x_2)\dot{x}_1 + (2x_1 + 2x_2)\dot{x}_2 \\
&= 4x_1x^2 + 2x_2^2 - 8(x_1 + x_2)^2 - 2(x_1 + x_2)h(x_1 + x_2) \\
&= -2x_1^2 - 6(x_1 + x_2)^2 - 2(x_1 + x_2)h(x_1 + x_2) \\
&\le -2x_1^2 - 6(x_1 + x_2)^2 \quad \forall \ |x_1 + x_2| \le 1 \\
&= -x^T \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} x < 0
\end{aligned}
$$

- Now we can find the domain $D = \{|x_1 + x_2| \le 1\}$ satisfying $\dot{V}(x) < 0$
- Furthermore, the invariant set $\Omega_c = \{V(x) \le c\}$ satisfying $\dot{V}(x) < 0$ is obtained by choosing

$$
c = \min_{|x_1+x_2|=1} x^T P x = \frac{1}{b^T P^{-1} b} = 1 \tag{40}
$$

because $b = [1, 1]^T$.

- Thus the set Ω_c with $c = 1$ is an estimate of the region of attraction.

5. Estimating the region of attraction by $\Omega_c = \{V(x) < c\}$ is simple, but usually very conservative. It can be extended by examining the region satisfying $\dot{V}(x) \le 0$.

(NC) 3.7 Converse Lyapunov Theorems

1. Theorems 3.3 and 3.6 establish asymptotic stability and exponential stability of the origin by requiring the existence of Lyapunov function $V(x)$ that satisfies certain conditions.

2. Converse Lyapunov Theorem would confirm that if the origin is asymptotically (or exponentially) stable, then $\exists\ V(x)$ that satisfies the conditions of Theorem 3.3 (or 3.6)

3. (Theorem 3.8) (Converse Lyapunov Theorem) Let $x = 0$ be an exponentially stable equilibrium point for $\dot{x} = f(x)$, where f is continuously differentiable on $D = \{\|x\| < r\}$. Let k, λ, and r_0 be positive constants with $r_0 < r/k \ni$

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \qquad \forall\ \ x(0) \in D_0 \quad \forall t \geq 0$$

   where $D_0 = \{\|x\| < r_0\}$. Then $\exists$ a continuously differentiable function $V(x)$ that satisfies the inequalities

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2 \qquad \frac{\partial V}{\partial x}f(x) \leq -c_3\|x\|^2 \qquad \left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|$$

   $\forall\ \ x \in D_0$, with positive constants c_1, c_2, c_3, and c_4.
   Moreover, if $D = D_0 = \Re^n$ and the origin is an exponentially stable equilibrium point, then $\exists$ $V(x)$ that satisfies the aforementioned inequalities $\forall\ x \in \Re^n$.

4. (Theorem 3.9) (Converse Lyapunov Theorem) Let $x = 0$ be an asymptotically stable equilibrium point for $\dot{x} = f(x)$, where f is locally Lipschitz on a domain $D \subset \Re^n$ that contains the origin. Let $R_A \subset D$ be the region of attraction of $x = 0$. Then, $\exists$ a smooth $V(x) > 0$ and a continuous $W(x) > 0$, both defined for all $x \in R_A \ni$

$$V(x) \to \infty \quad \text{as} \;\; x \to \partial R_A$$
$$\frac{\partial V}{\partial x} f(x) \leq -W(x) \quad \forall \;\; x \in R_A$$

and for any $c > 0$, $\{V(x) \leq c\}$ is a compact subset of R_A. When $R_A = \Re^n$, $V(x)$ is radially unbounded.

- (HW # 3) solve 5 problems 3.1, 3.5, 3.6, 3.10, 3.13 (if you want, 3.12)