(NC) 2 Two-Dimensional Systems

1. Why two-dimensional system?

   - mechanical system, electrical system, electromechanical system ...

2. Two-dimensional time-invariant systems occupy an important place in the study of nonlinear systems b/c solutions can be visualized by curves in the plane.

$$\dot{x}_1 = f_1(x_1, x_2) \qquad \dot{x}_2 = f_2(x_1, x_2) \tag{11}$$

3. Assume that f_1 and f_2 are locally Lipschitz over the domain $D \subset \Re^2$. Then the locus in the $x_1 - x_2$ plane of the solution $x = [x_1, x_2]^T \in \Re^2$, $\forall\, t \geq 0$, is a curve that passes through the point x_0 (initial condition).

4. (Phase Portrait and Vector Field) The family of all trajectories is called the phase portrait of (11). This $x_1 - x_2$ plane is called the phase plane. Using the vector notation, we rewrite (11) as

$$\dot{x} = f(x)$$

   where $f(x) = [f_1, f_2]^T$. The $f(x)$ is tangent to the trajectory at x and it is called a vector field on the phase plane.

5. (How to Draw a Vector Field) Consider

$$\dot{x}_1 = f_1(x_1, x_2) = 2x_1^2 \qquad\qquad \dot{x}_2 = f_2(x_1, x_2) = x_2$$

at $x = (1, 1)$, the vector field is obtained as the line segment from x to $x + f(x) = (3, 2)$ b/c

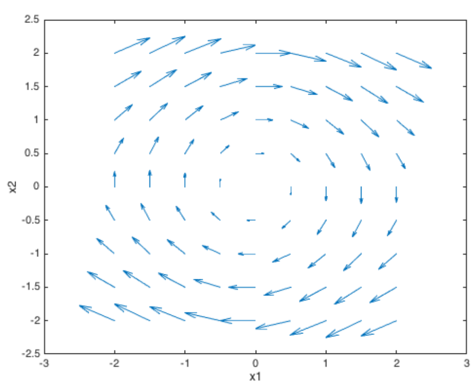
$$x_1 + f_1(x_1, x_2) = x_1 + 2x_1^2 = 3 \qquad\qquad x_2 + f_2(x_1, x_2) = 2x_2 = 2$$

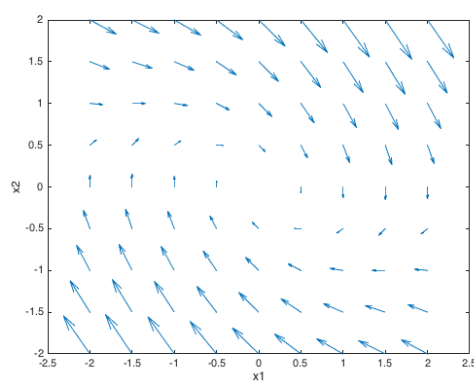
6. (Vector Field Diagram) The vector field at every point in a gird covering the plane is referred to as a vector field diagram, for example, draw a vector field diagram using the matlab

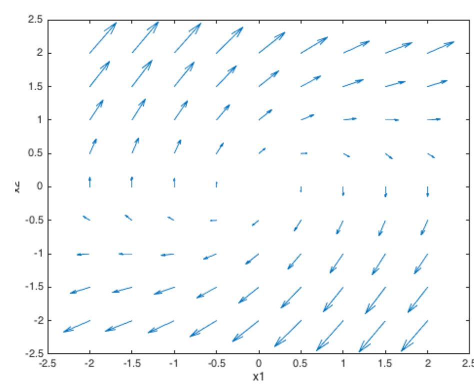
$$(1)\dot{x}_1 = x_2 \qquad (2)\dot{x}_1 = x_2 \qquad (3)\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\sin(x_1) \qquad \dot{x}_2 = -\sin(x_1) - x_2 \qquad \dot{x}_2 = -\sin(x_1) + x_2$$

```
% For visualization of vector field diagram
[x1, x2] = meshgrid(-2:0.5:2, -2:0.5:2);
x1dot = x2;   x2dot = -sin(x1);
quiver(x1,x2,x1dot,x2dot)
```







(NC) 2.1 Qualitative Behavior of Linear Systems (Diagonalization)

1. Consider the two-dimensional linear time-invariant system

$$\dot{x} = Ax \qquad \text{where } x \in \Re^2 \text{ and } A \in \Re^{2\times 2} \tag{12}$$

2. (Eigenvalue Decomposition) Let us obtain the eigenvalue decomposition of A in either real or complex number domain

$$Am_i = \lambda_i m_i \quad \rightarrow \quad AM = M\Lambda \quad \rightarrow \quad A = M\Lambda M^{-1} \quad \leftrightarrow \quad \Lambda = M^{-1}AM = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

for $i = 1, 2$, where $\lambda_1, \lambda_2 \in \Re$ or $\lambda_{1,2} = \alpha \pm j\beta \in \mathbb{C}$

3. (Example) Find the eigenvalue decomposition of the following matrix:

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

a) Let us find two eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = 0$$

$$\therefore \quad \lambda_1 = 1 + 2j \quad \text{and} \quad \lambda_2 = 1 - 2j$$

b) First eigenvector corresponding to $\lambda_1 = 1 + 2j$ is located on the null space $N(A - (1+2j)I)$.

$$(A - \lambda_1 I)m_1 = 0 \quad \rightarrow \quad \left[\begin{array}{cc|c} -2j & 2 & 0 \\ -2 & -2j & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{cc|c} j & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \therefore \quad m_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}$$

c) Second eigenvector corresponding to $\lambda_2 = 1-2j$ is located on the null space $N(A-(1-2j)I)$.

$$(A - \lambda_2 I)m_2 = 0 \quad \rightarrow \quad \begin{bmatrix} 2j & 2 & | & 0 \\ -2 & 2j & | & 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} j & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \quad \therefore \quad m_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix}$$

d) Eigenvalue decomposition using $M = [m_1; m_2]$ will be

$$\therefore \quad A = M\Lambda M^{-1} \quad \leftrightarrow \quad \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} 1+2j & 0 \\ 0 & 1-2j \end{bmatrix} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}^{-1}$$

4. The change of coordinates $z = M^{-1}x$ transforms the system into two decoupled scalar (one-dimensional) differential equations:

$$\dot{x} = Ax = M\Lambda M^{-1}x \quad \rightarrow \quad M^{-1}\dot{x} = \Lambda M^{-1}x \quad \rightarrow \quad \dot{z} = \Lambda z \quad \rightarrow \quad \dot{z}_1 = \lambda_1 z_1 \qquad \dot{z}_2 = \lambda_1 z_2 \tag{13}$$

with initial conditions $z_{10} = z_1(0)$ and $z_{20} = z_2(0)$, we have solutions:

$$\begin{aligned} \frac{dz_1}{z_1} &= \lambda_1 \, dt \\ \int_{z_1(0)}^{z_1(t)} \frac{dz_1}{z_1} &= \int_0^t \lambda_1 \, dt \\ \ln z_1(t) - \ln z_1(0) &= \lambda_1 \, t \\ \ln \frac{z_1(t)}{z_1(0)} &= \lambda_1 \, t \\ \frac{z_1(t)}{z_1(0)} &= e^{\lambda_1 t} \\ \therefore \qquad z_1(t) &= z_{10} e^{\lambda_1 t} \end{aligned}$$

$$\begin{aligned} \frac{dz_2}{z_2} &= \lambda_2 \, dt \\ \int_{z_2(0)}^{z_2(t)} \frac{dz_2}{z_2} &= \int_0^t \lambda_2 \, dt \\ \ln z_2(t) - \ln z_2(0) &= \lambda_2 \, t \\ \ln \frac{z_2(t)}{z_2(0)} &= \lambda_2 \, t \\ \frac{z_2(t)}{z_2(0)} &= e^{\lambda_2 t} \\ z_2(t) &= z_{20} e^{\lambda_2 t} \end{aligned}$$

(NC) 2.1 (Case 1. Real Eigenvalues)

1. For given two solutions, eliminating t between $z_1(t) = z_{10}e^{\lambda_1 t}$ and $z_2(t) = z_{20}e^{\lambda_2 t}$, we obtain

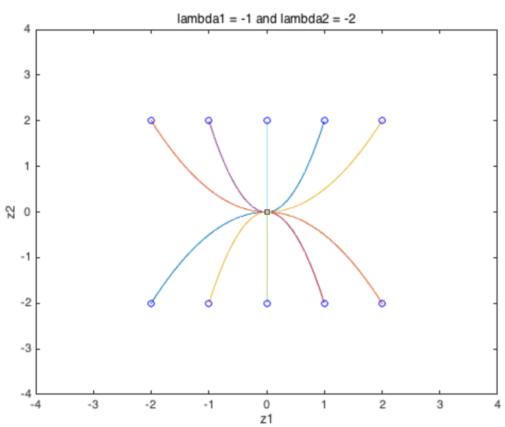
$$t = \frac{1}{\lambda_1}\ln\frac{z_1(t)}{z_{10}} \;\rightarrow\; z_2(t) = z_{20}e^{\frac{\lambda_2}{\lambda_1}\ln\frac{z_1(t)}{z_{10}}} \;\rightarrow\; z_2(t) = z_{20}e^{\ln\left(\frac{z_1(t)}{z_{10}}\right)^{\frac{\lambda_2}{\lambda_1}}} \;\rightarrow\; z_2 = \left(\frac{z_{20}}{z_{10}^{\frac{\lambda_2}{\lambda_1}}}\right)z_1^{\frac{\lambda_2}{\lambda_1}} \;\rightarrow\; z_2 = cz_1^{\frac{\lambda_2}{\lambda_1}}$$

2. (Stable Node) Let $\lambda_2 < \lambda_1 < 0$
   - $e^{\lambda_2 t} \rightarrow 0$ faster than $e^{\lambda_1 t}$, as $t \rightarrow \infty$.
   - $\frac{\lambda_2}{\lambda_1} > 1$, *e.g.*, if $\frac{\lambda_2}{\lambda_1} = 2$, then the curve of $z_2 = cz_1^2$ approaches to 0 as $t \rightarrow \infty$. [see figure 2.3]
   - The equilibrium point $z = 0$, *i.e.*, $x = 0$ is called stable node

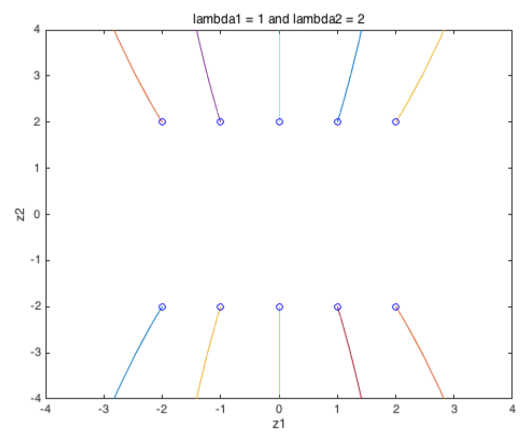
3. (Unstable Node) Let $\lambda_2 > \lambda_1 > 0$
   - $e^{\lambda_2 t} \rightarrow \infty$ faster than $e^{\lambda_1 t}$, as $t \rightarrow \infty$.
   - The figure (phase portrait) is the exactly same with the figure of $\lambda_2 < \lambda_1 < 0$, except the trajectory directions reversed
   - The equilibrium point $z = 0$, *i.e.*, $x = 0$ is called unstable node

4. (Saddle) Let $\lambda_2 < 0 < \lambda_1$
   - $e^{\lambda_2 t} \rightarrow 0$ but $e^{\lambda_1 t} \rightarrow \infty$, as $t \rightarrow \infty$.
   - $\frac{\lambda_2}{\lambda_1} < 0$, *e.g.*, if $\frac{\lambda_2}{\lambda_1} = -1$, then we have the curve of $z_2 = cz_1^{-1}$ [see figure 2.5]
   - The equilibrium point $z = 0$, *i.e.*, $x = 0$ is called saddle

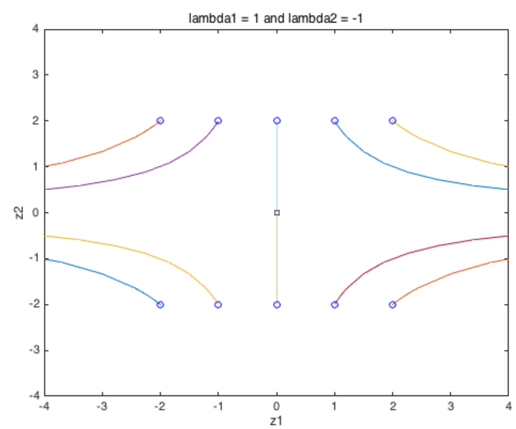
```
% For visualization of phase portrait for stable node
lambda1 = -1; lambda2 = -2;
f = @(t,z) [lambda1*z(1); lambda2*z(2)];
for z10 = [-2 -1 0 1 2]
   for z20 = [-2 2]
     [ts,ys] = ode45(f,[0,50],[z10;z20]);
     axis([-4 4 -4 4])
     plot(ys(:,1),ys(:,2)); hold on
     plot(ys(1,1),ys(1,2),'bo'); hold on % starting point denoted by circle
     plot(ys(end,1),ys(end,2),'ks') % ending point denoted by square
   end
end
```



stable node



unstable node



saddle

(NC) 2.1 (Case 2. Complex Eigenvalues)

1. If $\lambda_{1,2} = \alpha \pm j\beta \in \mathbb{C}$ with $\alpha, \beta \in \Re$ are applied to the solution, we have

$$z_1(t) = z_{10}e^{\lambda_1 t} = z_{10}e^{(\alpha+j\beta)t} = z_{10}e^{\alpha t}e^{j\beta t} = z_{10}e^{\alpha t}(\cos\beta t + j\sin\beta t)$$
$$z_2(t) = z_{20}e^{\lambda_2 t} = z_{20}e^{(\alpha-j\beta)t} = z_{20}e^{\alpha t}e^{-j\beta t} = z_{20}e^{\alpha t}(\cos\beta t - j\sin\beta t)$$

   thanks to Euler's Theorem $e^{j\theta} = \cos\theta + j\sin\theta$

2. Depending on the value of α, the trajectory will take one of the three forms

3. (Stable Focus) When $\alpha < 0$, the spiral converges to the origin. The equilibrium point is a stable focus [see figure 2.6(a)]

4. (Unstable Focus) When $\alpha > 0$, it diverges away from the origin. The equilibrium point is an unstable focus [see figure 2.6(b)]

5. (Center) When $\alpha = 0$, the trajectory is a circle of initial radius r_0. The equilibrium point is a center. [see figure 2.6(c)]. The system is vulnerable to perturbation when system matrix has eigenvalues on imaginary axis. (it is not robust if the equilibrium is a center)

6. (Hyperbolic Equilibrium) The equilibrium point is called hyperbolic if A has no eigenvalue with zero real part.

(NC) 2.2 Qualitative Behavior Near Equilibrium Points

1. Qualitative behavior of a nonlinear system near an equilibrium point can be determined via linearization w.r.t. that point

2. (Taylor Series Expansion) Let $x_0 = [x_{10}, x_{20}]^T$ be an equilibrium point and f_1 and f_2 are continuously differentiable. Expanding f_1 and f_2 into their Taylor series about x_0, we have

$$\dot{x}_1 = f_1(x_{10}, x_{20}) + a_{11}(x_1 - x_{10}) + a_{12}(x_2 - x_{20}) + H.O.T. \tag{14}$$

$$\dot{x}_2 = f_2(x_{10}, x_{20}) + a_{21}(x_1 - x_{10}) + a_{22}(x_2 - x_{20}) + H.O.T. \tag{15}$$

   where

   - $f_1(x_{10}, x_{20}) = f_2(x_{10}, x_{20}) = 0$
   - $a_{11} = \frac{\partial f_1}{\partial x_1}\Big|_{x=x_0}$, $a_{12} = \frac{\partial f_1}{\partial x_2}\Big|_{x=x_0}$, $a_{21} = \frac{\partial f_2}{\partial x_1}\Big|_{x=x_0}$ and $a_{22} = \frac{\partial f_2}{\partial x_2}\Big|_{x=x_0}$

3. (Linearized System at $x = x_0$) Let us introduce new definitions $y_1 = x_1 - x_{10}$ and $y_2 = x_2 - x_{20}$. If we restrict attention to a sufficiently small neighborhood of the equilibrium point such that the higher-order terms (H.O.T) are negligible, then we have

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 \tag{16}$$

$$\dot{y}_2 = a_{21}y_2 + a_{22}y_2 \tag{17}$$

   Rewriting the equation in a vector form gives $\dot{y} = Ay$, where $A = \frac{\partial f}{\partial x}\Big|_{x=x_0}$ is the Jacobian matrix of $f(x)$ evaluated at $x = x_0$.

4. It is true that if the origin of the linearized state equation is a stable (or unstable) node with distinct eigenvalues, saddle, a stable (or unstable) focus, then in a small neighborhood of the equilibrium point, the trajectories of the nonlinear state equation will behave like a stable (or unstable) node, saddle, a stable (or unstable) focus. [by Theorem 3.2]

5. If the origin of the linearized state equation is a center equilibrium, we cannot say that the equilibrium point of the nonlinear system is a center b/c it is vulnerable to the small perturbation. We should check it using nonlinear analysis

6. (Example 2.1) Find the properties of the origin of linearized system and the equilibrium point of nonlinear system ?

$$\dot{x}_1 = -x_2 - \mu x_1(x_1^2 + x_2^2) \qquad \dot{x}_2 = x_1 - \mu x_2(x_1^2 + x_2^2) \tag{18}$$

   - equilibrium point: $x = 0$
   - linearized state equation: $\dot{x} = Ax$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
   - origin of the linearized system is a center equilibrium b/c two eigenvalues are $\pm j$
   - in the polar coordinates, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, ($\dot{x}_1 = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$)

$$\dot{r} = -\mu r^3 \qquad \dot{\theta} = 1 \tag{19}$$

   - trajectories of the nonlinear system show
     - stable focus when $\mu > 0$
     - unstable focus when $\mu < 0$.

7. This example shows that the qualitative behavior describing a center in the linearized state equation is not preserved in the nonlinear state equation.

(NC) 2.3 Multiple Equilibria

1. The linear system $\dot{x} = Ax$ has
   - an isolated equilibrium point at $x = 0$, if $\det(A) \neq 0$
   - a continuum of equilibrium points located on the null space of A b/c $Ax = 0$, if $\det(A) = 0$.
2. A nonlinear system can have multiple isolated equilibrium points.
3. (Example 2.2) Find the equilibrium points and their property?

$$\dot{x}_1 = 0.5x_2 - 0.5h(x_1) \qquad \dot{x}_2 = -0.2x_1 - 0.3x_2 + 0.24 \tag{20}$$

   where $h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$

   - Equilibrium points are obtained from the intersection of $x_2 = h(x_1)$ and $x_2 = -\frac{2}{3}x_1 + 0.8$
   - Using matlab $roots(p)$ with $p = [83.72, -226.31, 229.62, -103.79, (17.76 + 2/3), -0.8]$, we can get 3 real roots and 2 complex roots. If we consider 3 real roots, then we have 3 equilibrium points as follows:

$$Q_1 = (0.0626, 0.7583) \quad Q_2 = (0.2854, 0.6097) \quad Q_3 = (0.8844, 0.2104)$$

   - Since Jacobian matrix of $f(x)$ is given by

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} -0.5(17.76 - 207.58x_1 + 688.86x_1^2 - 905.24x_1^3 + 418.6x_1^4) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}\Bigg|_{x=Q}$$

we have 3 system matrices evaluated at Q_1, Q_2, and Q_3

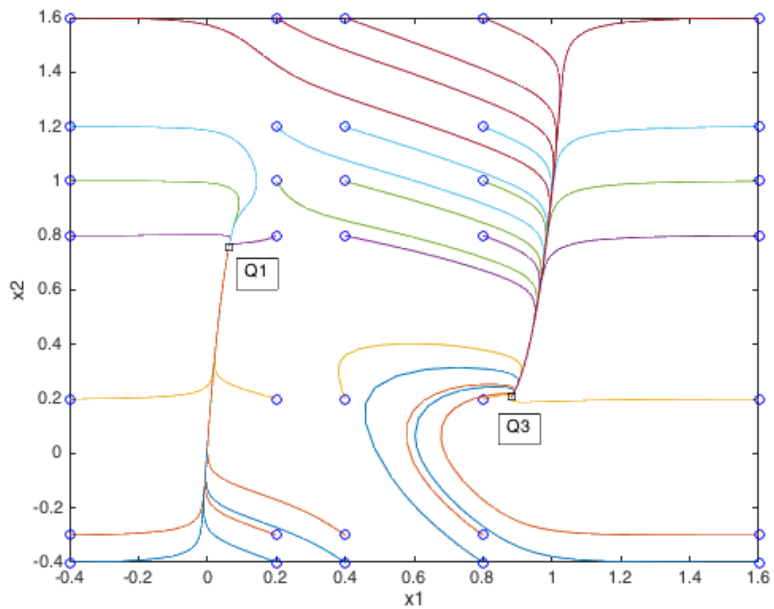
$$A_1 = \begin{bmatrix} -3.6247 & 0.5000 \\ -0.2000 & -0.3000 \end{bmatrix} \quad \lambda_1 = -3.5943 \text{ and } \lambda_2 = -0.3304 \quad Q_1 \text{ is a stable node}$$

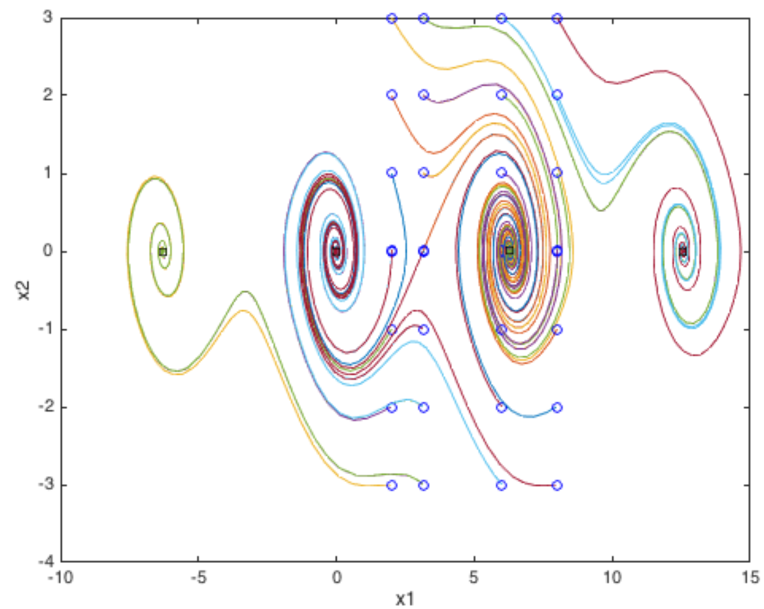
$$A_2 = \begin{bmatrix} 1.8201 & 0.5000 \\ -0.2000 & -0.3000 \end{bmatrix} \quad \lambda_1 = 1.7718 \text{ and } \lambda_2 = -0.2517 \quad Q_2 \text{ is a saddle}$$

$$A_3 = \begin{bmatrix} -1.4364 & 0.5000 \\ -0.2000 & -0.3000 \end{bmatrix} \quad \lambda_1 = -1.3402 \text{ and } \lambda_2 = -0.3961 \quad Q_3 \text{ is a stable node}$$

4. For visualization of phase portrait of Eq. (20), use the MATLAB

```
% For visualization of phase portrait
f = @(t,x) [0.5*x(2) - 0.5*(17.76*x(1) -103.79*x(1)^2+229.62*x(1)^3 ...
    -226.31*x(1)^4+83.72*x(1)^5); -0.2*x(1)-0.3*x(2)+0.24];
for x10 = [-0.4 0.2 0.4 0.8 1.6]
    for x20 = [-0.4 -0.3 0.2 0.8 1 1.2 1.6]
         [ts,ys] = ode45(f,[0,50],[x10;x20]);
         plot(ys(:,1),ys(:,2)); hold on
         plot(ys(1,1),ys(1,2),'bo'); hold on % starting points
         plot(ys(end,1),ys(end,2),'ks') % ending points
    end
end
```





5. (Example 2.3) Find the equilibrium points and their property?

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -\sin x_1 - 0.3x_2 \tag{21}$$

- Equilibrium points are obtained as $(n\pi, 0)$ from $\sin x_1 = 0$. Consider

$$Q_1 = (0, 0) \quad Q_2 = (\pi, 0)$$

- Since Jacobian matrix of $f(x)$ is given by

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -0.3 \end{bmatrix}\Bigg|_{x=Q}$$

we have 2 system matrices evaluated at Q_1 and Q_2

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -0.3 \end{bmatrix} \quad \lambda_{1or2} = -0.15 \pm j0.9887 \qquad Q_1 \text{ is a stable focus}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & -0.3 \end{bmatrix} \quad \lambda_1 = -1.1612 \text{ and } \lambda_2 = 0.8612 \qquad Q_2 \text{ is a saddle}$$

6. For visualization of phase portrait of Eq. (21), use the MATLAB

```
% For visualization of phase portrait
g = @(t,x) [x(2); -sin(x(1)) - 0.3*x(2) ];
for x10 = [2 pi pi+0.01 6 8]
    for x20 = [-3 -2 -1 -0.01 0.01 1 2 3 ]
        [ts,ys] = ode45(g,[0,50],[x10;x20]);
        plot(ys(:,1),ys(:,2)); hold on
        plot(ys(1,1),ys(1,2),'bo'); hold on % starting points
        plot(ys(end,1),ys(end,2),'ks') % ending points
    end
end
```

(NC) 2.4 Limit Cycles

1. Oscillation is one of the most important phenomena that occur in dynamical systems.
2. A system oscillates when it has a nontrivial periodic solution

$$x(t+T) = x(t), \qquad \forall\, t \geq 0 \tag{22}$$

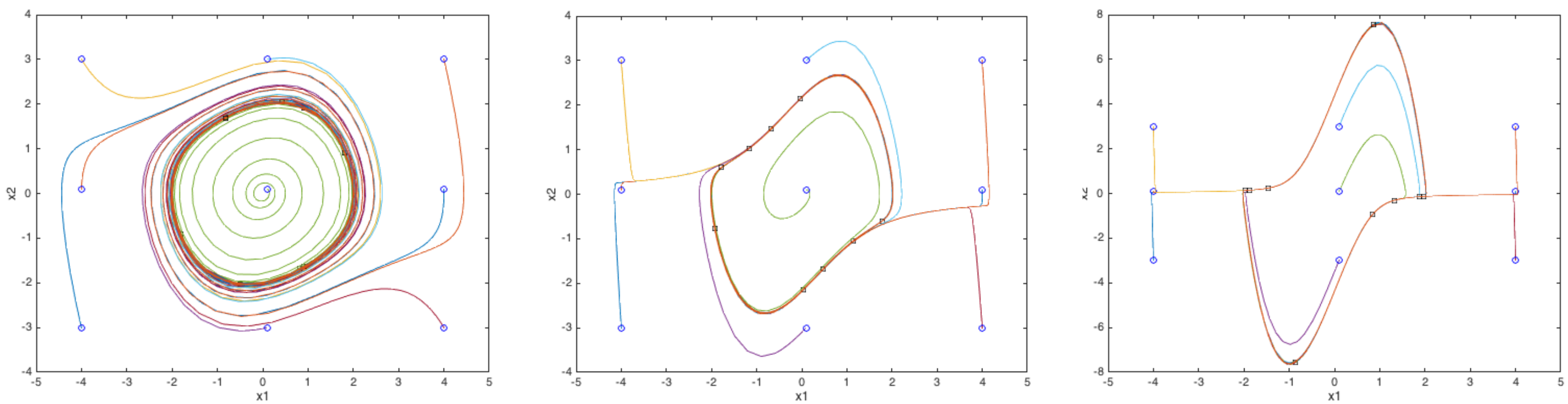
3. The image of a periodic solution in the phase portrait is a closed trajectory, periodic orbit, or closed orbit.
4. Linear oscillator
   - it is not structurally stable
   - the amplitude of oscillation is dependent on the initial conditions.
5. Nonlinear oscillator
   - it is structurally stable
   - the amplitude of oscillation is independent on the initial conditions.

6. (Example 2.4) Draw the phase portraits when $\epsilon = 0.2, 1.0, 5.0$

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2 \tag{23}$$

- Van der Pol oscillator, it has only one isolated closed orbit.
- When $\epsilon = 0.2$, the closed orbit is close to a circle of radius 2.
- When $\epsilon = 1.0$, the circular shape of the closed orbit is distorted
- When $\epsilon = 5.0$, the closed orbit is severely distorted.

```
epsilon = 0.2; v = @(t,x) [x(2); -x(1) + epsilon*(1-x(1)^2)*x(2) ];
for x10 = [-4 0.1 4]
    for x20 = [-3 0.1 3]
         [ts,ys] = ode45(v,[0,50],[x10;x20]);
         plot(ys(:,1),ys(:,2)); hold on
         plot(ys(1,1),ys(1,2),'bo'); hold on % starting points
         plot(ys(end,1),ys(end,2),'ks') % ending points
    end
end
```

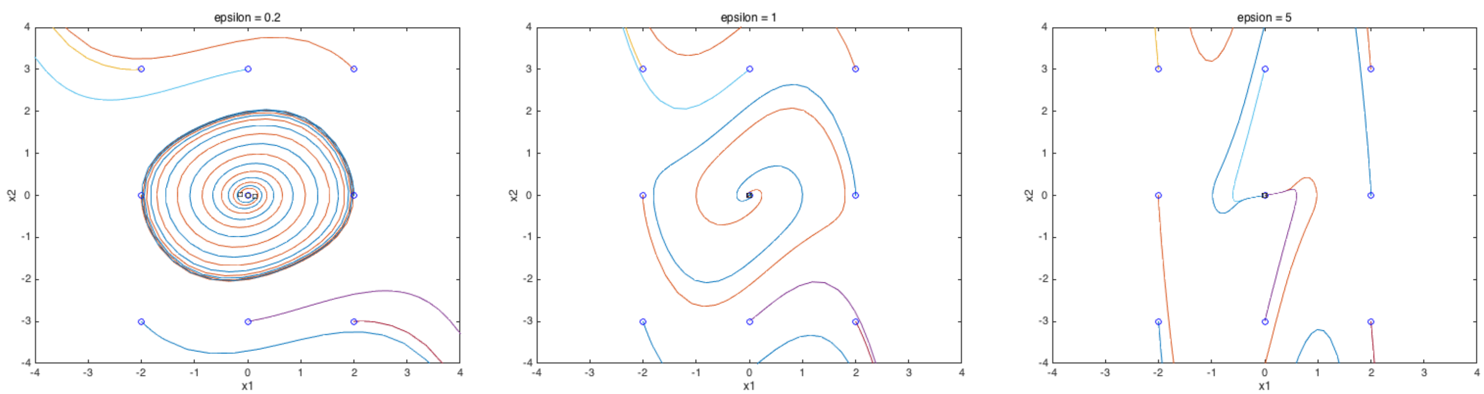


7. Like Van der Pol oscillator, if there is only one isolated closed orbit, it is called a limit cycle. Especially, since all the trajectories approach to it, then it is referred to as a stable limit cycle.

8. If all the trajectories close to limit cycle move away as time progress, it is called unstable limit cycle. Example of unstable limit cycle

$$\dot{x}_1 = -x_2 \qquad \dot{x}_2 = x_1 - \epsilon(1 - x_1^2)x_2$$

```
epsilon = 0.2; v = @(t,x) [-x(2); x(1) - epsilon*(1-x(1)^2)*x(2) ];
for x10 = [-2 0 2]
    for x20 = [-3 0 3]
         [ts,ys] = ode45(v,[0,50],[x10;x20]);
         axis([-4 4 -4 4])
         plot(ys(:,1),ys(:,2)); hold on
         plot(ys(1,1),ys(1,2),'bo'); hold on % starting point
         plot(ys(end,1),ys(end,2),'ks') % ending point
    end
end
```



- (HW # 2) solve 5 problems 2.1, 2.2, 2.4, 2.7. and 2.10.