

# 제 9 장

## Nonlinear Systems

### 1 Why Study Nonlinear Systems?

- Kochenburger - Describing function in 1950
  - A describing function can be defined for nonlinear elements, including those with memory.
- Lyapunov - stability of motion in 1892
  1. First method : if the linear approximation is strictly stable (having all roots in the left half-plane), then the nonlinear system will have a region of stability around the equilibrium point where the linear approximation applies. Furthermore, if the linear approximation has at least one root in the right half-plane, then the nonlinear system cannot have any region of stability in the neighborhood of the equilibrium. (effectiveness of linearization)
  2. Second method : if Lyapunov function (internal energy stored in the system,  $V > 0$  and  $\dot{V} < 0$ ) can be found, then the system on which it is based will be stable.
- If the internal energy decreases according to the passage of time, the system will be stable. (concept of stability)

## 2 Analysis by Linearization

- (9.2.1) Linearization by Small-Signal Analysis

1. Assume that nonlinear system is given by

$$\dot{x} = f(x, u)$$

2. Equilibrium point is defined as the state vector  $x = x_0 \in \mathfrak{R}^n$  and input  $u = u_0 \in \mathfrak{R}$  satisfying  $\dot{x} = 0$  such as

$$\dot{x} = 0 = f(x_0, u_0)$$

3. Let us apply Taylor series expansion about the equilibrium point  $x = x_0$  and  $u = u_0$

$$\dot{x} = f(x_0, u_0) + \left. \frac{\partial f(x, u)}{\partial x^T} \right|_{x=x_0, u=u_0} (x - x_0) + \left. \frac{\partial f(x, u)}{\partial u} \right|_{x=x_0, u=u_0} (u - u_0) + H.O.T$$

4. Let us define new states as follows:

$$\bar{x} = x - x_0$$

$$\bar{u} = u - u_0$$

then we have

$$\dot{\bar{x}} + \dot{x}_0 = f(x_0, u_0) + \left. \frac{\partial f(x, u)}{\partial x^T} \right|_{x=x_0, u=u_0} \bar{x} + \left. \frac{\partial f(x, u)}{\partial u} \right|_{x=x_0, u=u_0} \bar{u} + H.O.T$$

where

$$\dot{x}_0 = f(x_0, u_0)$$

5. Now we have an approximate system as follow:

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

where

$$A = \left. \frac{\partial f(x, u)}{\partial x^T} \right|_{x=x_0, u=u_0} \quad B = \left. \frac{\partial f(x, u)}{\partial u} \right|_{x=x_0, u=u_0}$$

where  $A$  and  $B$  are the best linear fits to the nonlinear function  $f(x, u)$  at  $x_0$  and  $u_0$ .

6. As a result, the control input is designed as

$$u = u_0 + \bar{u}$$

where  $\bar{u}$  can be designed according to the linear control theories.

7. For above procedures, we assumed either that the motion was small or that motion from some operating point was small, so that nonlinear functions were approximated by linear functions.

8. (Example 9.1) Check the stability of the pendulum system at the equilibrium points

– Equation of motion

$$ml^2\ddot{\theta} + mgl \sin \theta = T_c \quad \rightarrow \quad \ddot{\theta} + \frac{g}{l} \sin \theta = \frac{T_c}{ml^2} \quad \rightarrow \quad \ddot{\theta} + \omega_o^2 \sin \theta = u$$

where  $\omega_o = \sqrt{\frac{g}{l}}$  and  $u = \frac{T_c}{ml^2}$ .

– Nonlinear differential equation

$$x_1 = \theta$$

$$x_2 = \dot{\theta}$$

$$\dot{x}_1 = \dot{\theta} = x_2$$

$$\dot{x}_2 = \ddot{\theta} = -\omega_o^2 \sin x_1 + u$$

Thus we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\omega_o^2 \sin x_1 + u \end{bmatrix} \quad \leftarrow \quad \dot{x} = f(x, u) \in \mathbb{R}^2$$

– Equilibrium points

$$\dot{x} = 0 \quad \rightarrow \quad x_2 = 0 \quad \text{and} \quad \sin x_1 = 0 \quad \text{and} \quad u = 0 \quad \rightarrow \quad x_1 = n\pi \quad \text{and} \quad x_2 = 0 \quad \text{and} \quad u = 0$$

where  $n$  is an integer. In other words,

$$x_0 = \begin{bmatrix} n\pi \\ 0 \end{bmatrix} \quad \text{and} \quad u_0 = 0$$

- The linearized system at the equilibrium points  $\bar{x} = x - x_0$  and  $\bar{u} = u - u_0$

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$A = \left. \frac{\partial f(x, u)}{\partial x^T} \right|_{x=x_0, u=u_0} = \left[ \begin{array}{cc} \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} \\ \frac{\partial(-\omega_o^2 \sin x_1)}{\partial x_1} & \frac{\partial(-\omega_o^2 \sin x_1)}{\partial x_2} \end{array} \right] \bigg|_{x=x_0, u=u_0} = \left[ \begin{array}{cc} 0 & 1 \\ -\omega_o^2 \cos x_1 & 0 \end{array} \right] \bigg|_{x=x_0, u=u_0}$$

$$B = \left. \frac{\partial f(x, u)}{\partial u} \right|_{x=x_0, u=u_0} = \left[ \begin{array}{c} \frac{\partial 0}{\partial u} \\ \frac{\partial u}{\partial u} \end{array} \right] \bigg|_{x=x_0, u=u_0} = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]$$

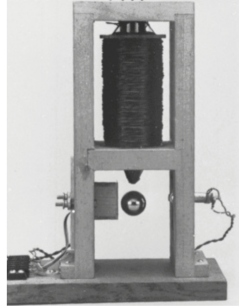
- At  $x_0 = [0, 0]$  and  $u_0 = 0$ , the equilibrium is *neutrally stable* because

$$A = \left[ \begin{array}{cc} 0 & 1 \\ -\omega_o^2 & 0 \end{array} \right] \quad \text{eigenvalues} \quad \pm j\omega_o$$

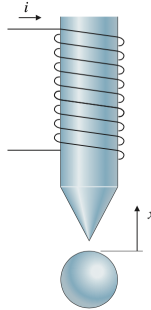
- At  $x_0 = [\pi, 0]$  and  $u_0 = 0$ , the equilibrium is *unstable* because

$$A = \left[ \begin{array}{cc} 0 & 1 \\ \omega_o^2 & 0 \end{array} \right] \quad \text{eigenvalues} \quad \pm \omega_o$$

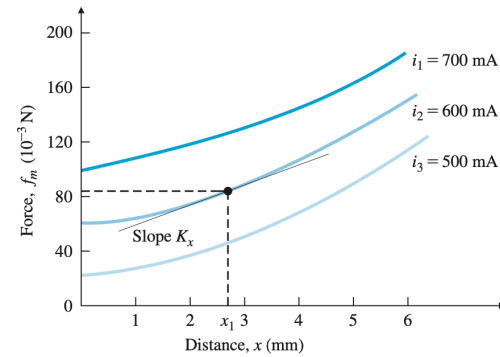
**Figure 9.1**  
Magnetic ball levitator  
used in the laboratory  
Source: Photo courtesy of Gene  
Franklin



**Figure 9.2**  
Model for ball  
levitation



**Figure 9.3**  
Experimentally  
determined force curves



9. (Example 9.2) Linearize the following nonlinear equation using the experimental result shown in Fig. 9.3 at the equilibrium  $x = x_1$  and  $i = i_2 = 0.6A$

a) Equation of motion

$$m\ddot{x} = f_m(x, i) - mg$$

where  $f_m(x, i)$  obtained as the Fig. 9.3, and  $mg = 0.082N$  and  $m = 0.0084kg$

b) Linearization of  $f_m(x, i)$  with  $\bar{x}_1 = x - x_1$  and  $\bar{i} = i - i_2$

$$\begin{aligned} f_m(x_1 + \bar{x}_1, i_2 + \bar{i}) &\approx f_m(x_1, i_2) + K_x \bar{x}_1 + K_i \bar{i} \\ &= 0.082 + 14\bar{x}_1 + 0.4\bar{i} \end{aligned}$$

where

$$K_x = 14N/m$$

$$K_i = \frac{0.122 - 0.042}{0.7 - 0.5} = 0.4N/A$$

c) The linearized system at the equilibrium points

$$\begin{aligned}m(\ddot{\bar{x}}_1 + \ddot{\bar{x}}_1) &\approx f_m(x_1, i_2) + K_x \bar{x}_1 + K_i \bar{i} - mg \\0.0084\ddot{\bar{x}}_1 &= (0.082 + 14\bar{x}_1 + 0.4\bar{i}) - 0.082 \\&= 14\bar{x}_1 + 0.4\bar{i} \\\ddot{\bar{x}}_1 &= 1667\bar{x}_1 + 47.6\bar{i}\end{aligned}$$

d) Let us assign new state  $\bar{x}_2 = \dot{\bar{x}}_1$  and control  $u = \bar{i}$ , then we have

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1667 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 47.6 \end{bmatrix} u$$

where  $x = \bar{x}_1 + x_1[m]$  and  $i = u + i_2 = u + 0.6[A]$

- (9.2.2) Linearization by Nonlinear Feedback

1. By *subtracting* the nonlinear terms out of the dynamic equations and *adding* them to the control, the nonlinear system can be a linear system.
2. For robots with two or three rigid links, it is called *computed torque approach*.
3. (Example 9.4) Linearize the system by using the nonlinear feedback

$$ml^2\ddot{\theta} + mgl \sin \theta = T_c$$

If we compute the torque to be

$$T_c = mgl \sin \theta + u$$

then the motion is described by

$$ml^2\ddot{\theta} = u$$

In addition, the auxiliary control  $u$  can be designed as

$$u = -K_v\dot{\theta} + K_p(\theta_t - \theta) \quad \rightarrow \quad T_c = mgl \sin \theta - K_v\dot{\theta} + K_p(\theta_t - \theta)$$

where  $K_v$  and  $K_p$  are gains and  $\theta_t$  is a target to be tracked.



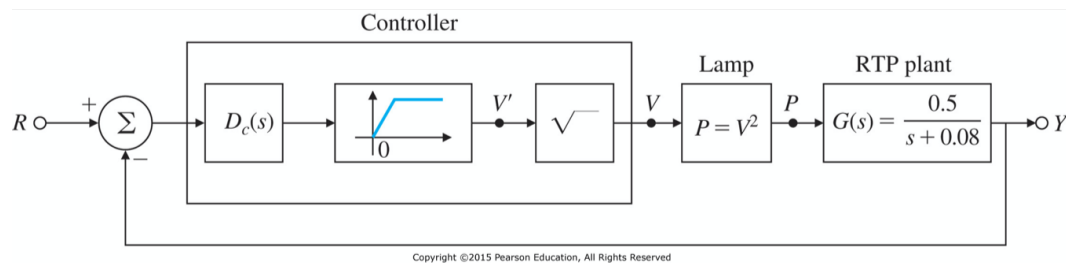
- (9.2.3) Linearization by Inverse Nonlinearity

1. It is sometimes possible to reverse the effect of some nonlinearities. For example, suppose we have a system whose output is the square of the signal of interest:

$$y = x^2$$

One clever and rather obvious technique is to undo the nonlinearity by preceding the physical nonlinearity with the square root nonlinearity

$$x = \sqrt{(\cdot)}$$



2. (Example 9.5) Consider the RTP (rapid thermal processing) system that uses a nonlinear lamp as an actuator as shown in Fig. 9.5. If we apply the following nonlinearity

$$V = \sqrt{V'} \quad \rightarrow \quad Y = G(s)P = G(s)V^2 = G(s)V'$$

Thus we can use *linear control* design techniques for the dynamic compensator  $D_c(s)$ .