2. Transfer Function (TF) and Frequency Response

- Definition of Laplace Transform (LT) for any causal LTI system with $t \geq 0$

$$\mathcal{L}[y(t)] = Y(s) = \int_0^\infty y(t)e^{-st}dt$$

If we apply the LT to convolution integral, we have by using $t - \tau = \eta$:

$$\begin{aligned}
Y(s) &= \int_0^\infty \left[\int_0^\infty u(t-\tau)h(\tau)d\tau\right] e^{-st}dt \\
&= \int_0^\infty \left[\int_0^\infty u(t-\tau)e^{-st}dt\right] h(\tau)d\tau \\
&= \int_0^\infty \left[\int_{-\tau}^\infty u(\eta)e^{-s(\tau+\eta)}d\eta\right] h(\tau)d\tau \\
&= \int_0^\infty \left[\int_0^\infty u(\eta)e^{-s\eta}d\eta\right] h(\tau)e^{-s\tau}d\tau \\
&= \left[\int_0^\infty h(\tau)e^{-s\tau}d\tau\right] \left[\int_0^\infty u(\eta)e^{-s\eta}d\eta\right] \\
&= H(s)U(s)
\end{aligned}$$

where $U(s)$ is the LT of input signal and $H(s)$ is the LT of impulse response,

- Transfer Function (TF) of the system is defined as the LT of impulse response of the system
- TF is the ratio between LT of input and LT of output signals assuming zero initial conditions.

$$H(s) = \frac{Y(s)}{U(s)}$$

- It is noted that the convolution integral is replaced by a simple multiplication of the LT.
- Consider LT of the unit impulse function

$$\mathcal{L}[\delta(t)] = \int_0^\infty \delta(t)e^{-st}dt = \int_{0-}^{0+} \delta(t)dt = 1$$

Thus the TF of the system $H(s)$ is equal to the LT of the impulse response b/c $U(s) = 1$

$$H(s) = Y(s) \qquad \text{when the input has the form of unit impulse}$$

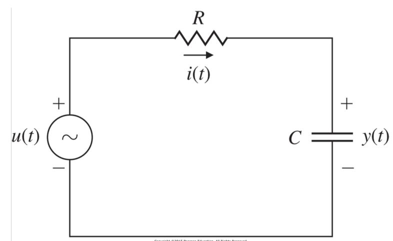
- LT of the differentiation

$$\begin{aligned}
\mathcal{L}[\dot{y}(t)] &= \int_0^\infty \dot{y}(t)e^{-st}dt = \int_0^\infty e^{-st}\dot{y}(t)dt \\
&= e^{-st}y(t)\Big|_0^\infty - (-s)\int_0^\infty e^{-st}y(t)dt \\
&= 0 - y(0) + s\int_0^\infty y(t)e^{-st}dt \\
&= sY(s) - y(0)
\end{aligned}$$

(Example 3.4, TF) Compute the TF of $\dot{y} + ky = u$ with zero initial conditions.

$$\mathcal{L}[\dot{y} + ky] = \mathcal{L}[u] \quad \rightarrow \quad \mathcal{L}[\dot{y}] + k\mathcal{L}[y] = \mathcal{L}[u] \quad \rightarrow \quad sY(s) - y(0) + kY(s) = U(s)$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+k}$$



- (Example 3.5, TF of RC Circuit) Compute the TF of RC circuit

$$Ri(t) + y(t) = u(t) \quad \text{and} \quad i(t) = C\frac{dy(t)}{dt}$$

Take the LT with zero initial condition

$$RI(s) + Y(s) = U(s) \quad \text{and} \quad I(s) = C(sY(s) - y(0)) = CsY(s)$$

Then we have

$$RCsY(s) + Y(s) = U(s) \quad \rightarrow \quad H(s) = \frac{Y(s)}{U(s)} = \frac{1}{RCs+1}$$

- LT of real exponential function e^{-at}

$$
\begin{aligned}
\mathcal{L}[e^{-at}] &= \int_0^\infty e^{-at} e^{-st} dt \\
&= \int_0^\infty e^{-(s+a)t} dt \\
&= -\frac{1}{s+a} e^{-(s+a)t} \Big|_0^\infty \\
&= \frac{1}{s+a}
\end{aligned}
$$

- LT of complex exponential function $e^{-j\omega t}$

$$
\begin{aligned}
\mathcal{L}[e^{-j\omega t}] &= \int_0^\infty e^{-j\omega t} e^{-st} dt \\
&= \int_0^\infty e^{-(j\omega+s)t} dt \\
&= -\frac{1}{s+j\omega} e^{-(s+j\omega)t} \Big|_0^\infty \\
&= \frac{1}{s+j\omega}
\end{aligned}
$$

- Euler theorem ($e^{j\theta} = \cos\theta + j\sin\theta$) and sinusoidal (cosine and sine) functions

$$
\begin{aligned}
e^{j\omega t} &= \cos\omega t + j\sin\omega t \\
e^{-j\omega t} &= \cos\omega t - j\sin\omega t
\end{aligned}
$$

By summing the above and dividing by half, we have the definition of cosine function

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

By subtracting the above and dividing by $2j$, we have the definition of sine function

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

- LT of the sinusoidal functions

$$
\begin{aligned}
\mathcal{L}[\cos\omega t] &= \frac{1}{2}\left\{\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right\} \\
&= \frac{1}{2}\frac{2s}{s^2+\omega^2} \\
&= \frac{s}{s^2+\omega^2} \\
\mathcal{L}[\sin\omega t] &= \frac{1}{2j}\left\{\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right\} \\
&= \frac{1}{2j}\frac{2j\omega}{s^2+\omega^2} \\
&= \frac{\omega}{s^2+\omega^2}
\end{aligned}
$$

(Example 3.6, 3.7, Complete response (=transient + steady state response)) Obtain the output of the system $H(s) = \frac{1}{s+1}$ when the input $u(t) = \sin 10t$ for $t \geq 0$ is applied ?

$$U(s) = \mathcal{L}[\sin 10t] = \frac{10}{s^2 + 100}$$

The output response is

$$\begin{aligned}
Y(s) = H(s)U(s) &= \frac{1}{s+1}\frac{10}{s^2+100} \\
&= \frac{a}{s+1} + \frac{bs+c}{s^2+100} \qquad \text{with three unknowns } a, b, c \\
&= \frac{10}{101}\left\{\frac{1}{s+1} + \frac{-s+1}{s^2+100}\right\} \\
&= \frac{10}{101}\left\{\frac{1}{s+1} - \frac{s}{s^2+100} + \frac{1}{s^2+100}\right\} \\
&= \frac{10}{101}\frac{1}{s+1} - \frac{10}{101}\frac{s}{s^2+100} + \frac{1}{101}\frac{10}{s^2+100}
\end{aligned}$$

Take the inverse LT

$$y(t) = \frac{10}{101}e^{-t} - \frac{10}{101}\cos 10t + \frac{1}{101}\sin 10t \qquad \text{for } t \geq 0$$

- Frequency response (main topic of chapter 6) is defined as the steady-state response when the sinusoidal input is applied. In other words, the transient responses are ignored.
- From previous example, the frequency response can be obtained by ignoring the transient response as follow:

$$\begin{aligned} y_{ss}(t) &= \frac{1}{101}\sin 10t - \frac{10}{101}\cos 10t \\ &= \frac{1}{\sqrt{101}}\sin(10t - \tan^{-1} 10) \\ &= \frac{1}{\sqrt{101}}\sin(10t - 84.3^\circ) \qquad \text{for } t \geq 0 \end{aligned}$$

  where $A\sin\omega t + B\cos\omega t = \sqrt{A^2+B^2}\sin(\omega t + \phi)$ and $\phi = \tan^{-1}\frac{B}{A}$.
- Indeed, steady-state response due to the sinusoidal input $u(t) = A\sin\omega t$ is obtained as

$$y_{ss}(t) = A|H(j\omega)|\sin(\omega t + \angle H(j\omega))$$

  This means that if a system represented by the TF $H(s)$ has a sinusoidal input with magnitude A, the output will be sinusoidal at the same frequency with magnitude $A|H(j\omega)|$ and will be shifted in phase by the angle $\angle H(j\omega)$, where $|H(j\omega)|$ is call as magnitude ratio and $\angle H(j\omega)$ as phase difference.

- **For example, if $H(s) = \frac{1}{s+1}$ and $u(t) = \sin 10t$, then the magnitude ratio and the phase difference at a specific frequency $\omega = 10[rad/s]$ are**

$$|H(j\omega)| = \left|\frac{1}{j\omega + 1}\right| = \frac{1}{\sqrt{\omega^2 + 1}} = \frac{1}{\sqrt{101}}$$

$$\angle H(j\omega) = \angle\frac{1}{j\omega + 1} = -\tan^{-1}\omega = -\tan^{-1}10 = -84.3^\circ$$

we have the frequency response as follow:

$$u(t) = \sin 10t \qquad \rightarrow \qquad y_{ss}(t) = \frac{1}{\sqrt{101}}\sin(10t - 84.3^\circ)$$

along with

$$u(t) = A\sin\omega t \qquad \rightarrow \qquad y_{ss}(t) = A|H(j\omega)|\sin(\omega t + \angle H(j\omega))$$

3. The $\mathcal{L}_-$ Laplace Transform (LT)

- One-sided (or unilateral) LT with the complex variable $s = \sigma + j\omega$

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

where it is noted that $\mathcal{L}_+$ Laplace Transform (LT) is defined

$$F(s) = \int_{0+}^{\infty} f(t)e^{-st}dt = \int_{0-}^{\infty} f(t)e^{-st}dt - \int_{0-}^{0+} f(t)e^{-st}dt = \int_{0-}^{\infty} f(t)e^{-st}dt - \int_{0-}^{0+} f(t)dt$$

- Two-sided LT

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

- Inverse LT

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}ds$$

where σ_c is a selected value to the right of all the singularities of $F(s)$ in the s-plane.

(Example 3.8, Step and Ramp) Find the LT of the step $a \cdot 1(t)$ and ramp $bt \cdot 1(t)$

$$
\begin{aligned}
F_1(s) &= \int_0^\infty ae^{-st}dt \\
&= -\frac{ae^{-st}}{s}\bigg|_0^\infty \\
&= 0 - \frac{-a}{s} \\
&= \frac{a}{s} \\
F_2(s) &= \int_0^\infty bte^{-st}dt \\
&= -\frac{bte^{-st}}{s}\bigg|_0^\infty - \int_0^\infty -\frac{be^{-st}}{s}dt \\
&= -\frac{bte^{-st}}{s}\bigg|_0^\infty + -\frac{be^{-st}}{s^2}\bigg|_0^\infty \\
&= \frac{b}{s^2}
\end{aligned}
$$

- see Table A.2 in Appendix A (page 867)