3 The Three-Term Controller: PID Control

- Through long experience and by trial and error, we have discovered "integral control" as a means of eliminating bias offset.
- In case of poor dynamic response, an "anticipatory term" based on the derivative was added. The result is called PID controller and has the TF

$$D_c(s) = k_P + \frac{k_I}{s} + k_D s$$

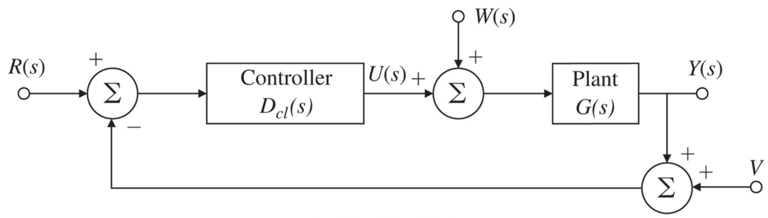
where k_P is the "proportional gain", k_I is the "integral gain", and k_D is the "derivative gain".

1. Proportional Control (P)
   - When the feedback control signal is linearly proportional to the system error,

   $$u(t) = k_P e(t)$$

   - The TF of P control

   $$\frac{U(s)}{E(s)} = D_c(s) = k_P$$



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- For example, consider Fig. 4.2 with $G(s) = \frac{1}{s^2+a_1s+a_2}$ and $D_c(s) = k_P$, the characteristic equation becomes

$$1 + \frac{k_P}{s^2 + a_1 s + a_2} \qquad \rightarrow \qquad s^2 + a_1 s + (a_2 + k_P) \qquad \rightarrow \qquad s^2 + 2\zeta\omega_n s + \omega_n^2$$

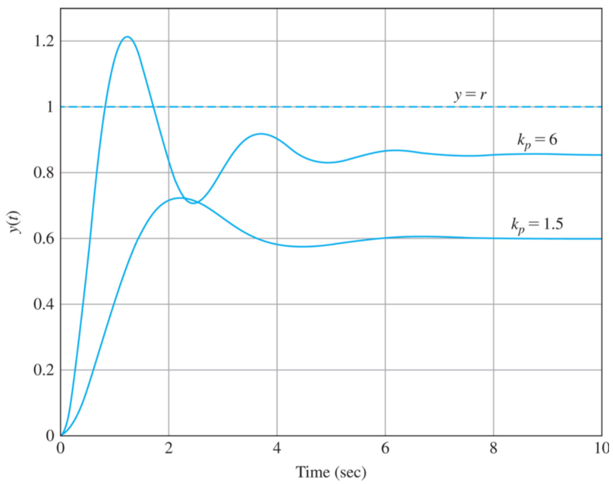
  a) Here, we can determine the natural frequency using the k_P, but we cannot control the damping term a_1 since it is independent of k_P.

$$\omega_n = \sqrt{a_2 + k_P} \qquad\qquad \zeta = \frac{a_1}{2\sqrt{a_2 + k_P}}$$

  b) The system is Type 0 because the steady-state error remains in case of step input.

$$\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{k_P G(s)}{1 + k_P G(s)} = \frac{k_P}{s^2 + a_1 s + (a_2 + k_P)} \\
y_{ss} &= \lim_{s\to 0} sY(s) \\
&= \lim_{s\to 0} s\frac{k_P}{s^2 + a_1 s + (a_2 + k_P)}\frac{1}{s} = \frac{k_P}{a_2 + k_P} \\
e_{ss} &= 1 - y_{ss} = \frac{a_2}{a_2 + k_P}
\end{aligned}$$

  where the error decreases and the response exhibits a decrease in damping as the gain increases. (Fig. 4.7)



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c) The error due to a disturbance is given by

$$T_w(s) = -\frac{Y(s)}{W(s)} = -\frac{G(s)}{1 + k_P G(s)} = -\frac{1}{s^2 + a_1 s + (a_2 + k_P)} = s^0 T_{o,w}(s) \quad \rightarrow \quad \text{type 0}$$

$$e_{ss} = -\lim_{s \to 0} s \frac{1}{s^2 + a_1 s + (a_2 + k_P)} \frac{1}{s} = -\frac{1}{a_2 + k_P} \quad \rightarrow \quad K_{0,w} = -(a_2 + k_P)$$

where the error due to the step disturbance decreases as the gain increases.

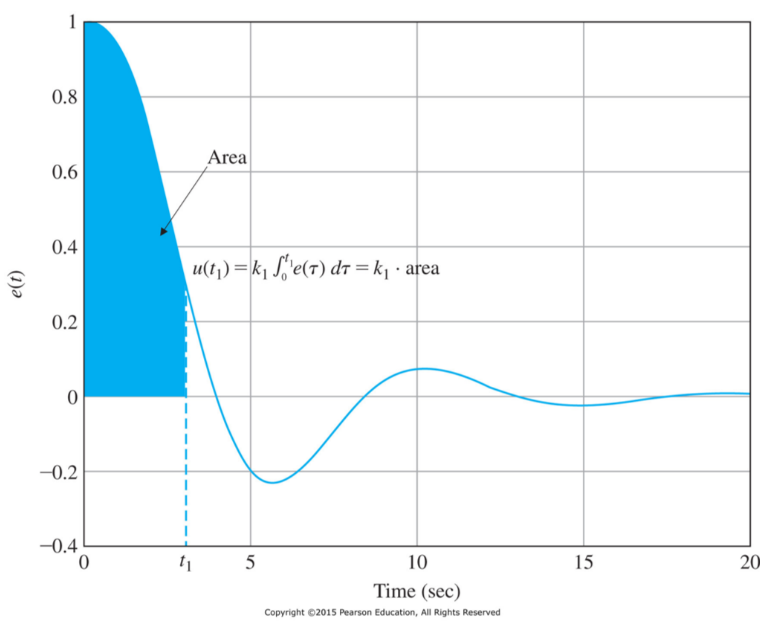
- For systems beyond second order, the situation is more complicated than that illustrated above. A higher gain will increase the speed of response but typically at the cost of a larger transient overshoot and less overall damping
- One way to improve the steady-state accuracy of control without using extremely high proportional gain is to introduce integral control

2. Integral Control (I)

- When the feedback control signal is linearly proportional to the integral of the system error,

$$u(t) = k_I \int_0^t e(\tau) d\tau$$

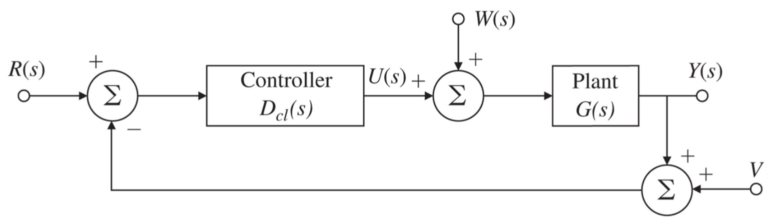
- The goal of integral control is to minimize the steady-state tracking error and the steady-state output response to disturbances.
- This means that the control signal at each instant of time is a summation of all past values of the tracking error, therefore, the control action is based on the history of the system error.



- Fig. 4.8 illustrates that the control signal at any instant of time is proportional to the area under the system error curve.

- The TF of I control

$$\frac{U(s)}{E(s)} = D_c(s) = \frac{k_I}{s}$$



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- Again, consider the Fig. 4.2 under the integral control.

$$\frac{Y(s)}{R(s)} = \frac{k_I G(s)}{s + k_I G(s)} \qquad \frac{E(s)}{R(s)} = \frac{s}{s + k_I G(s)}$$

- Assume unit-step reference input of $R(s) = \frac{1}{s}$ and unit DC gain of $G(0) = 1$, the steady-state errors

$$y_{ss} = \lim_{s \to 0} s \frac{k_I G(s)}{s + k_I G(s)} \frac{1}{s} = 1$$
$$e_{ss} = \lim_{s \to 0} s \frac{s}{s + k_I G(s)} \frac{1}{s} = 0$$

- Since the system of $D_c(s)G(s) = \frac{k_I G(s)}{s}$ with $G(0) = 1$ is type 1, thus we have the velocity constant as follow:

$$K_v = k_I \qquad \rightarrow \qquad e_{ss} = \frac{1}{k_I} \quad \text{for ramp input}$$

$$= \lim_{s \to 0} s \frac{s}{s + k_I G(s)} \frac{1}{s^2} = \frac{1}{k_I}$$

- The integral gain k_I can be selected purely to provide an acceptable dynamic response; however, typically it will cause instability if raised sufficiently high.
- The error due to a disturbance is given by ($R(s) = 0$)

$$T_w(s) = -\frac{Y(s)}{W(s)} = -\frac{sG(s)}{s + k_I G(s)} = s^1 T_{o,w}(s) \qquad \rightarrow \qquad \text{type 1}$$

$$e_{ss} = -\lim_{s \to 0} s \frac{sG(s)}{s + k_I G(s)} \frac{1}{s} = 0 \qquad \text{for step disturbance}$$

$$e_{ss} = -\lim_{s \to 0} s \frac{sG(s)}{s + k_I G(s)} \frac{1}{s^2} = -\frac{1}{k_I} \qquad \rightarrow \qquad K_{1,w} = -k_I \qquad \text{for ramp disturbance}$$

- In conclusion, the integral feedback results in zero steady-state output error (type 1) in both stepwise tracking and disturbance rejection. Plant parameter changes can be tolerated, the results above are independent of the plant parameter values. These properties of integral control are referred to as “robust”.

3. Derivative Control (D)

- The goal of derivative feedback is

  (1) to improve the closed-loop system stability
  (2) speeding up the transient response
  (3) reducing overshoot.

  Whenever increased stability is desired, the use of derivative feedback is called for.

- The control law is

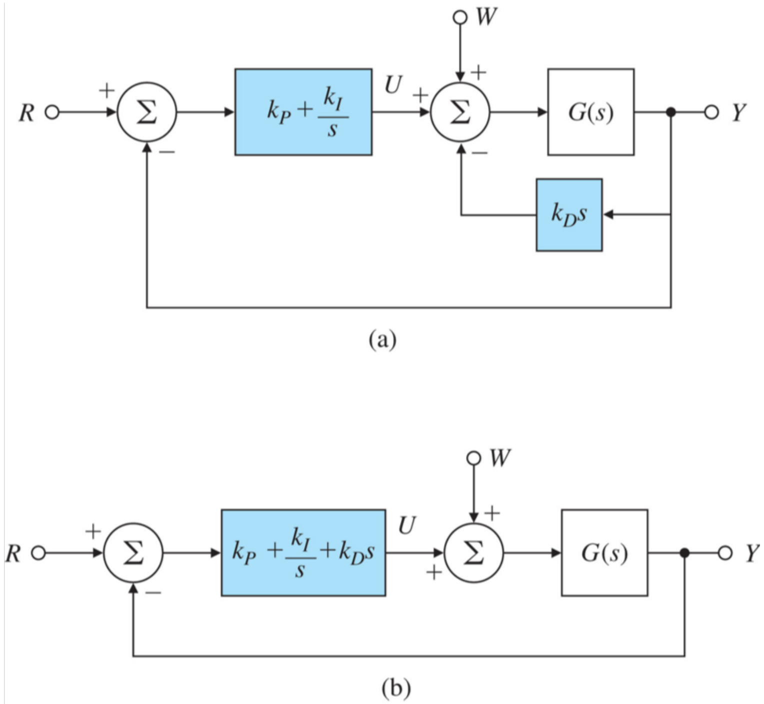
$$u(t) = k_D \dot{e}(t)$$

  where the derivative control is almost never used by itself; it is usually augmented by proportional control.

- The TF of D control

$$\frac{U(s)}{E(s)} = D_c(s) = k_D s$$

- A key feature is that derivative control knows the slope of the error signal, so it takes control action based on the trend in the error signal. Hence it is said to have an "anticipatory behavior".
- One disadvantage of derivative control is that it tends to amplify noise.



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- See the Fig. 4.10, both configurations result in the same characteristic equation (poles), but the the zeros from the reference to the output are different in the both cases.
- With the derivative in the feedback path as shown in Fig. 4.10(a), the reference is not differentiated, which is how the undesirable response to sudden changes is avoided.

4. Proportional plus Integral Control (PI)

- Adding an integral term to the proportional controller to achieve the lower steady-state errors results in the PI control equation in the time domain:

$$u(t) = k_P e(t) + k_I \int_0^t e(\tau) d\tau$$

- The TF of PI control

$$\frac{U(s)}{E(s)} = D_c(s) = k_P + \frac{k_I}{s}$$

- Introduction of the integral term raises the type to Type 1 and the system can therefore reject completely constant bias disturbance.
- solve (Example 4.5)

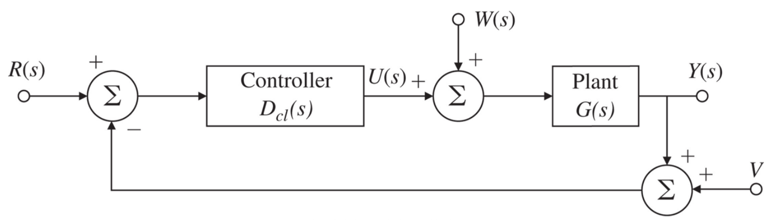
5. PID Control

- PID control equation in the time domain

$$u(t) = k_P e(t) + k_I \int_0^t e(\tau) d\tau + k_D \dot{e}(t)$$

- The TF of PID control

$$\frac{U(s)}{E(s)} = D_c(s) = k_P + \frac{k_I}{s} + k_D s$$



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- For example, consider Fig. 4.2 with $G(s) = \frac{A}{s^2+a_1s+a_2}$ and $D_c(s) = k_P + \frac{k_I}{s} + k_D s$. The characteristic equation becomes

$$1 + G(s)D_c(s) = 0 \qquad \rightarrow \qquad 1 + \frac{A}{s^2 + a_1 s + a_2}(k_P + \frac{k_I}{s} + k_D s) = 0$$

$$s^3 + (a_1 + Ak_D)s^2 + (a_2 + Ak_P)s + Ak_I = 0$$

where three roots (poles) are selected arbitrary in theory.

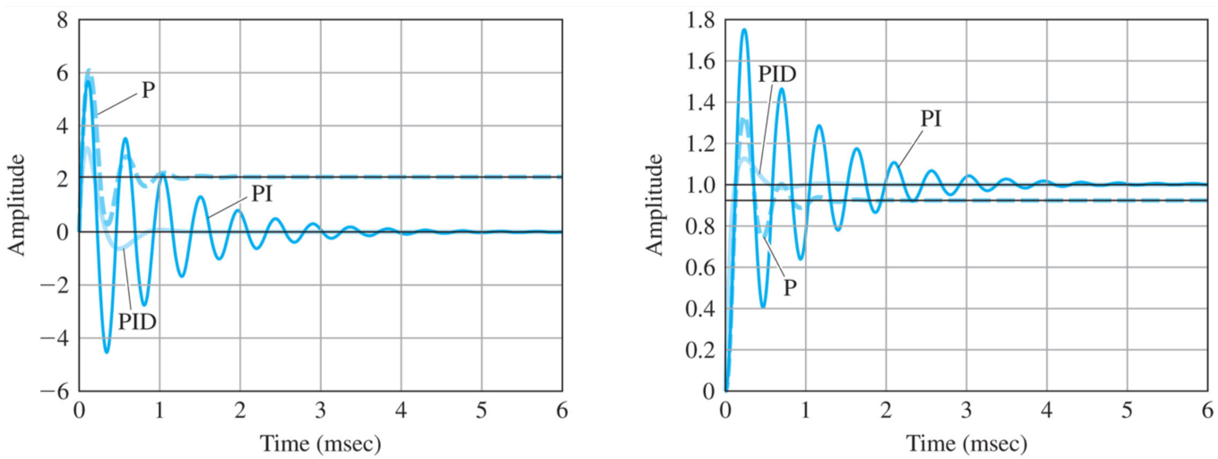
- (Example 4.6 PID Control of Motor Speed) Consider the result of Example 2.15

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{K_t}{s[(J_m s + b)(L_a s + R_a) + K_t K_e]}$$

If we apply the P, PI, and PID controllers, respectively, then we have the response curves shown in Fig. 4.16. Please check it using MATLAB.

  – Adding the integral term increases the oscillatory behavior but eliminates the steady-state error

  – Adding the derivative term reduces the oscillation while maintaining zero steady-state error.



(a) (b)

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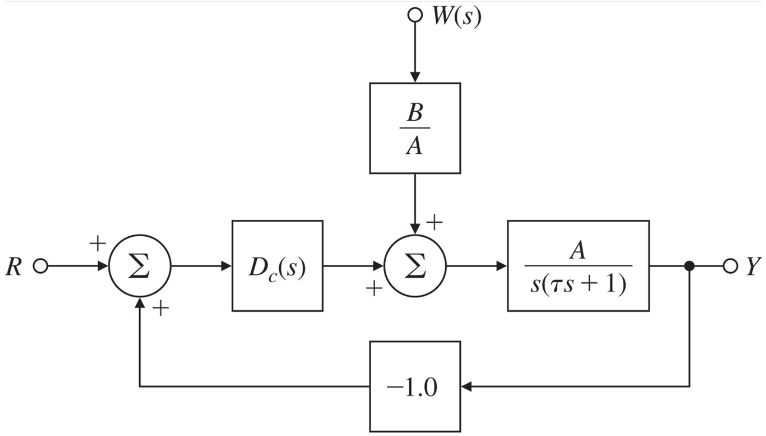
- solve (Example 4.7, PI Control for a DC-DC Voltage Converter)
- solve (Example 4.8, Cone Displacement Control for a Loudpeaker)



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- (Example PI Control for a DC Motor Position Control) Consider the Fig. 4.6. Determine the system type and the steady-state error with respect to disturbance input. ($R(s) = 0$) (a) when P control is applied,

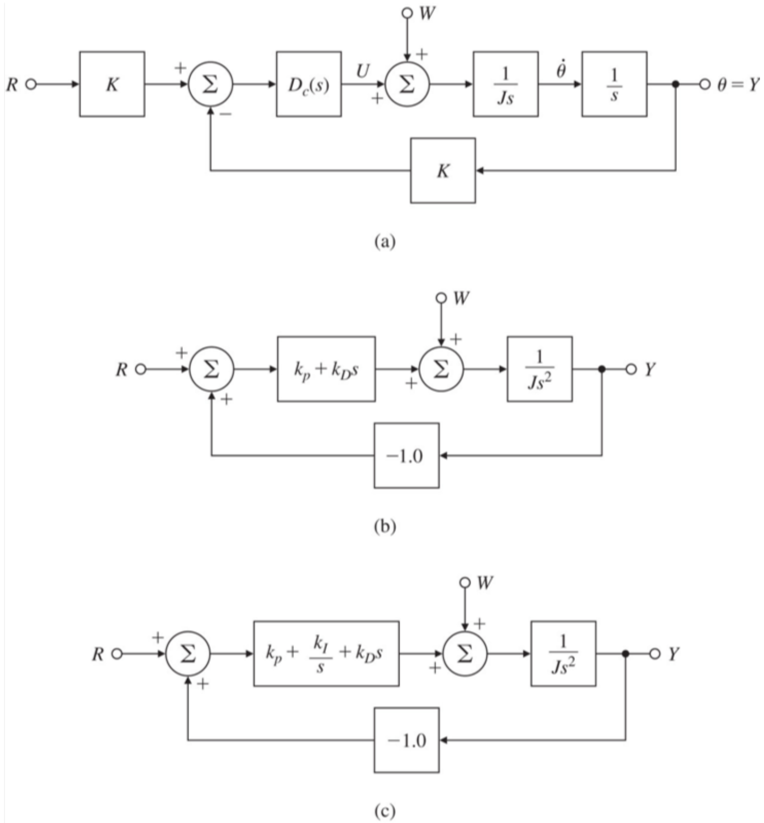
$$
\begin{aligned}
T_w(s) = \frac{E(s)}{W(s)} &= -\frac{B}{s(\tau s + 1) + Ak_p} \\
&= -\frac{B}{\tau s^2 + s + Ak_p} \\
&= s^0 T_{o,w}(s) \quad \rightarrow \quad \text{type 0} \\
e_{ss} &= -\frac{B}{Ak_P} \quad \text{for step disturbance} \quad \rightarrow \quad K_{0,w} = -\frac{Ak_p}{B}
\end{aligned}
$$

(b) when PI control is applied

$$
\begin{aligned}
T_w(s) &= -\frac{Bs}{s^2(\tau s+1)+A(k_p s+k_I)} \\
&= -\frac{Bs}{\tau s^3+s^2+Ak_p s+Ak_I} \\
&= s^1 T_{o,w}(s) \qquad \rightarrow \qquad \textbf{type 1}
\end{aligned}
$$

$$
e_{ss}=0 \qquad \textbf{for step disturbance}
$$

$$
e_{ss}=-\frac{B}{Ak_I} \qquad \textbf{for ramp disturbance} \qquad \rightarrow \qquad K_{1,w}=-\frac{Ak_I}{B}
$$



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- (Example Satellite Attitude Control) For given figure (b), determine the system types and error response to disturbance

$$T_w(s) = \frac{E(s)}{W(s)} = -\frac{\frac{1}{Js^2}}{1 + \frac{k_P + k_D s}{Js^2}} = -\frac{1}{Js^2 + k_D s + k_P}$$

$$= s^0 T_{o,w}(0) \qquad \rightarrow \qquad \text{type } 0$$

$$e_{ss} = -\frac{1}{k_P} \quad \text{for step disturbance} \qquad \rightarrow \qquad K_{0,w} = -k_P$$

For given the figure (c),

$$
\begin{aligned}
T_w(s) = \frac{E(s)}{W(s)} &= -\frac{\frac{1}{Js^2}}{1 + \frac{k_P s + k_D s^2 + k_I}{Js^3}} = -\frac{s}{Js^3 + k_D s^2 + k_P s + k_I} \\
&= s^1 T_{o,w}(s) \qquad \rightarrow \qquad \text{type 1}
\end{aligned}
$$

$$e_{ss} = 0 \quad \text{for step disturbance}$$

$$e_{ss} = -\frac{1}{k_I} \quad \text{for ramp disturbance} \qquad \rightarrow \qquad K_{1,w} = -k_I$$