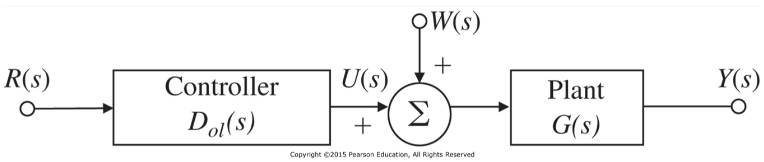
제 4 장

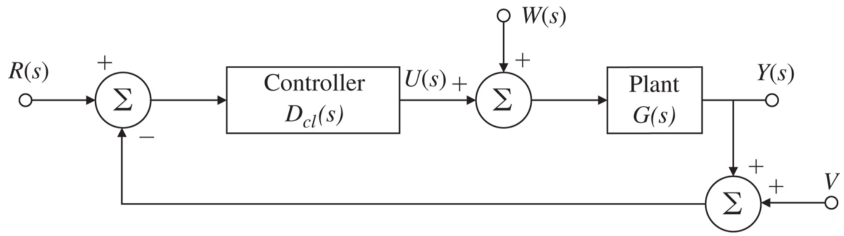
A First Analysis of Feedback

1 Basic Equations of Control



- For the open-loop control system of Fig. 4.1, the output, the error, and open-loop TF $\frac{Y_{ol}(s)}{R(s)}$ have the following forms:

$$
\begin{aligned}
Y_{ol}(s) &= G(s)D_{ol}(s)R(s) + G(s)W(s) \\
E_{ol}(s) &= R(s) - Y_{ol}(s) \\
&= R(s) - G(s)D_{ol}(s)R(s) - G(s)W(s) \\
&= [1 - G(s)D_{ol}(s)]R(s) - G(s)W(s) \\
T_{ol}(s) &= \frac{Y_{ol}(s)}{R(s)} = G(s)D_{ol}(s)
\end{aligned}
$$



- For the closed-loop control system of Fig. 4.2,

$$Y_{cl}(s) = \frac{G(s)D_{cl}(s)}{1+G(s)D_{cl}(s)}R(s) + \frac{G(s)}{1+G(s)D_{cl}(s)}W(s) - \frac{G(s)D_{cl}(s)}{1+G(s)D_{cl}(s)}V(s)$$

$$\begin{aligned} E_{cl}(s) &= R(s) - Y_{cl}(s) \\ &= \frac{1}{1+G(s)D_{cl}(s)}R(s) - \frac{G(s)}{1+G(s)D_{cl}(s)}W(s) + \frac{G(s)D_{cl}(s)}{1+G(s)D_{cl}(s)}V(s) \end{aligned}$$

For simplicity, let us define two TFs as follows:

$$S = \frac{1}{1+GD_{cl}} \qquad T = \frac{GD_{cl}}{1+GD_{cl}}$$

Using above two TFs, we can rewrite the output and error

$$Y_{cl} = TR + GSW - TV$$

$$E_{cl} = SR - GSW + TV$$

With these equations, we will explore four basic objectives of stability, tracking, regulation, and sensitivity for both the open-loop and closed-loop cases.

1. Stability

- Fundamental requirement for the control system is that all the poles of the TF should be located in the left half-plane (LHP)
- For the open-loop case with $G(s) = \frac{b(s)}{a(s)}$ and $D_{ol}(s) = \frac{c(s)}{d(s)}$

$$G(s)D_{ol}(s) = \frac{b(s)c(s)}{a(s)d(s)}$$

it cannot be stable if either $a(s)$ or $d(s)$ may have roots in the right half-plane (RHP)

- For the closed-loop case with $G(s) = \frac{b(s)}{a(s)}$ and $D_{cl}(s) = \frac{c(s)}{d(s)}$, the characteristic equation becomes

$$1 + G(s)D_{cl}(s) = 0 \qquad \rightarrow \qquad 1 + \frac{b(s)c(s)}{a(s)d(s)} = 0 \qquad \rightarrow \qquad a(s)d(s) + b(s)c(s) = 0$$

unlike the open-loop case, having a pole of $a(s)$ in the RHP does not prevent the design of a feedback controller that will make the system stable.

- For example (Governor problem by Maxwell), $G(s) = \frac{1}{s^2-1}$ and $D_{cl} = \frac{K(s+\gamma)}{s+\delta}$,

$$(s^2 - 1)(s + \delta) + K(s + \gamma) = 0 \qquad \rightarrow \qquad s^3 + \delta s^2 + (K - 1)s + K\gamma - \delta = 0$$

Using Routh stability criterion:

$$\begin{array}{ll} s^3 : 1 & (K-1) \\ s^2 : \delta & (K\gamma - \delta) \\ s^1 : (K-1) - \dfrac{(K\gamma - \delta)}{\delta} & \\ s^0 : K\gamma - \delta & \end{array}$$

the stability conditions are obtained as follows:

$$K > \frac{\delta}{\gamma} \quad \textbf{and} \quad \delta > \gamma > 0$$

Simple solution is to take $\gamma = 1$ (stable cancellation) and then the resultant second-order system can be easily solved to place the remaining two poles at any point desired.

$$s^2 + (\delta - 1)s + (K - \delta) = 0 \qquad \rightarrow \qquad s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

For the analysis of the control performance,

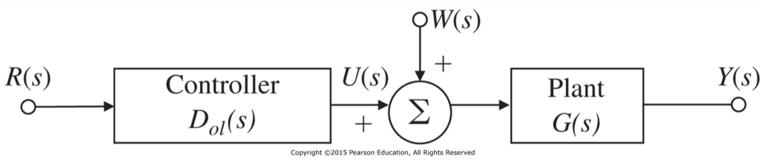
$$\omega_n = \sqrt{K - \delta} \qquad \zeta = \frac{\delta - 1}{2\sqrt{K - \delta}}$$

On the contrary, for the design of controller

$$\delta = 2\zeta\omega_n + 1 \qquad K = \omega_n^2 + 2\zeta\omega_n + 1$$

2. Tracking

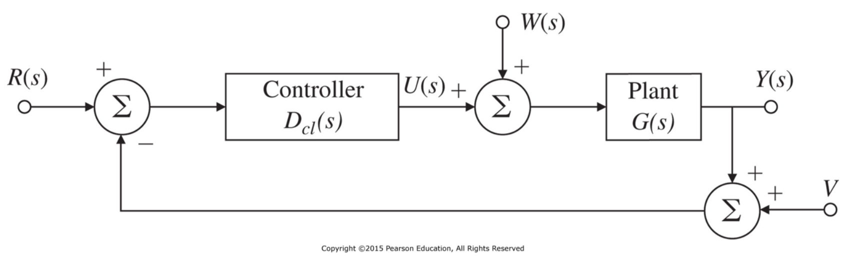
- Tracking is to cause the output to follow the reference input as closely as possible.



- In the open-loop case,

$$D_{ol}(s) = \frac{1}{G(s)}$$

  a) In order to physically build it, the controller TF must be proper
  b) The engineer must not get greedy and request an unrealistically fast design
  c) Although one can stably cancel any pole in the LHP, the sensitivity can be bad.



- For example, with $G(s) = \frac{1}{s^2+3s+9}$ and $D_{cl}(s) = \frac{c_2 s^2 + c_1 s + c_0}{s(s+d_1)}$, solve for the parameters of this controller so that the closed-loop will have the characteristic equation $(s+6)(s+3)(s^2+3s+9)$

$$
\begin{aligned}
1 + G(s)D_{cl}(s) &= 0 \\
1 + \frac{1}{s^2+3s+9}\frac{c_2 s^2 + c_1 s + c_0}{s(s+d_1)} &= 0 \\
(s^2+3s+9)s(s+d_1) + c_2 s^2 + c_1 s + c_0 &= (s+6)(s+3)(s^2+3s+9) \\
(s^2+3s+9)(s^2+d_1 s+c_2) &= (s+6)(s+3)(s^2+3s+9) \qquad \text{with } \frac{c_1}{c_2} = 3 \qquad \frac{c_0}{c_2} = 9
\end{aligned}
$$

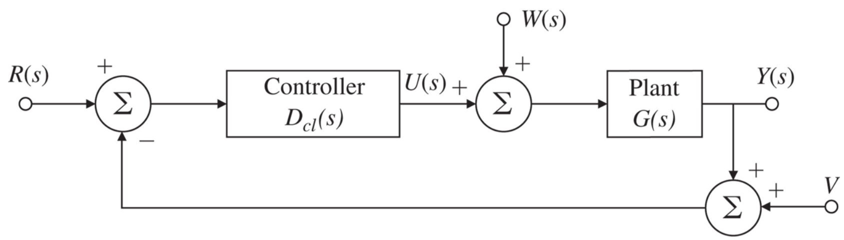
where $d_1 = 9$, $c_2 = 18$, $c_1 = 54$ and $c_0 = 162$

- For example, find the steady-state error for the step response with magnitude A

$$
\begin{aligned}
Y_{cl}(s) &= \frac{G(s)D_{cl}(s)}{1+G(s)D_{cl}(s)}R(s) \\
&= \frac{\frac{1}{s^2+3s+9}\frac{c_2s^2+c_1s+c_0}{s(s+d_1)}}{1+\frac{1}{s^2+3s+9}\frac{c_2s^2+c_1s+c_0}{s(s+d_1)}}\frac{A}{s} \\
&= \frac{c_2 s^2 + c_1 s + c_0}{s^4 + (3+d_1)s^3 + (9+3d_1+c_2)s^2 + (9d_1+c_1)s + c_0}\frac{A}{s} \\
y_{cl}(\infty) &= \lim_{s\to 0} sY_{cl}(s) = A \\
e_{cl}(\infty) &= A - y_{cl}(\infty) = 0
\end{aligned}
$$

3. Regulation

- Regulation is to keep the error small when the reference is at most a constant set-point and disturbances are present.



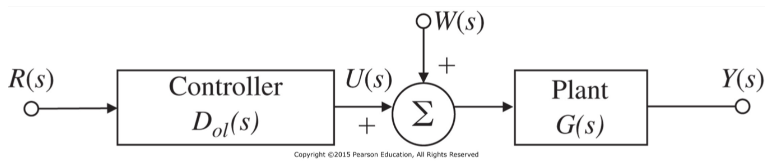
- In the closed-loop case, the error TF is described by

$$E_{cl}(s) = \frac{1}{1+G(s)D_{cl}(s)}R(s) - \frac{G(s)}{1+G(s)D_{cl}(s)}W(s) + \frac{G(s)D_{cl}(s)}{1+G(s)D_{cl}(s)}V(s)$$

- Consider the TF from the disturbance to the error $\frac{G(s)}{1+G(s)D_{cl}(s)}$, we should make $D_{cl}(s)$ as large as possible, to make the disturbance effect small.
- Consider the TF from the sensor noise to the error $\frac{G(s)D_{cl}(s)}{1+G(s)D_{cl}(s)}$, if we select $D_{cl}(s)$ to be large, it tends to unity and sensor noise is not reduced at all.
- Most disturbances exist at low frequency and Sensor noise has high frequency components. Using this information, we design the controller TF to be large at the low frequencies to reduce the disturbance effect and we will make it small at the higher frequencies to reduce the sensor noise effect.

4. Sensitivity

- Assume that the plant changes $G(s) \to G(s) + \delta G(s)$ in operation.



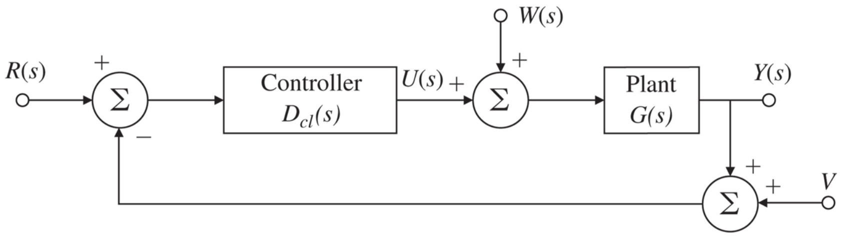
- In open-loop case,

$$T_{ol}(s) + \delta T_{ol}(s) = D_{ol}(G(s) + \delta G(s)) \qquad \to \qquad T_{ol} = D_{ol}G(s) \qquad \delta T_{ol}(s) = D_{ol}\delta G(s)$$

The sensitivity of a TF $T_{ol}(s)$ to a plant $G(s)$ is defined to be the ratio of the fractional change in $T_{ol}(s)$ to the fractional change in $G(s)$

$$S_G^{T_{ol}} = \frac{\frac{\delta T_{ol}}{T_{ol}}}{\frac{\delta G}{G}} = \frac{G}{T_{ol}}\frac{\delta T_{ol}}{\delta G} = \frac{1}{D_{ol}}D_{ol} = 1$$

where it means that 10% error in G would yield 10% error in T_{ol} in the open-loop case.



- In closed-loop case, δT_{cl} is complex to be handled.

$$T_{cl} + \delta T_{cl} = \frac{D_{cl}(G + \delta G)}{1 + D_{cl}(G + \delta G)} \quad \rightarrow \quad T_{cl} = \frac{D_{cl}G}{1 + D_{cl}G} \quad \delta T_{cl} = \frac{D_{cl}(G + \delta G)}{1 + D_{cl}(G + \delta G)} - T_{cl} \quad (?)$$

On the other hand, the first order variation is proportional to the derivative

$$\frac{\delta T_{cl}}{\delta G} = \frac{dT_{cl}}{dG} = \frac{D_{cl}(1 + D_{cl}G) - D_{cl}GD_{cl}}{(1 + D_{cl}G)^2} = \frac{D_{cl}}{(1 + D_{cl}G)^2}$$

The sensitivity of a TF $T_{cl}(s)$ to a plant $G(s)$ is found as following form:

$$S_G^{T_{cl}} = \frac{\frac{\delta T_{cl}}{T_{cl}}}{\frac{\delta G}{G}} = \frac{G}{T_{cl}}\frac{\delta T_{cl}}{\delta G} = \frac{G}{T_{cl}}\frac{dT_{cl}}{dG} = \frac{1 + D_{cl}G}{D_{cl}}\frac{D_{cl}}{(1 + D_{cl}G)^2} = \frac{1}{1 + D_{cl}G}$$

For example, if the gain is such that $1 + GD_{cl} = 100$, 10% change in plant gain G will cause only a 0.1% change in the steady-state gain.

- In feedback control, the error in overall TF gain is less sensitive to variations in the plant gain by a factor of $S = \frac{1}{1+GD_{cl}}$ compared to errors in open-loop control gain.
- Sensitivity function for a feedback system is defined as

$$S \triangleq \frac{1}{1 + GD_{cl}}$$

  In addition, the complementary sensitivity function (equal to the closed-loop TF) is defined as

$$T \triangleq 1 - S = \frac{GD_{cl}}{1 + GD_{cl}}$$

  Also it is marked in mind that

$$S + T = 1$$