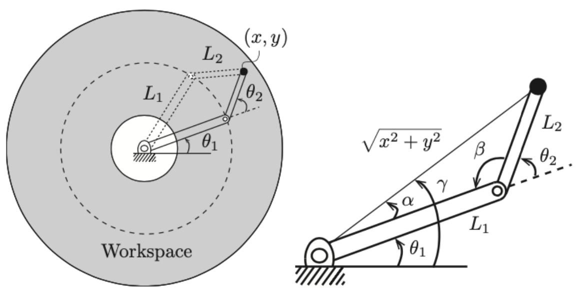
제 6 장

Inverse Kinematics (IK)



(a) A workspace, and lefty and righty configurations.

(b) Geometric solution.

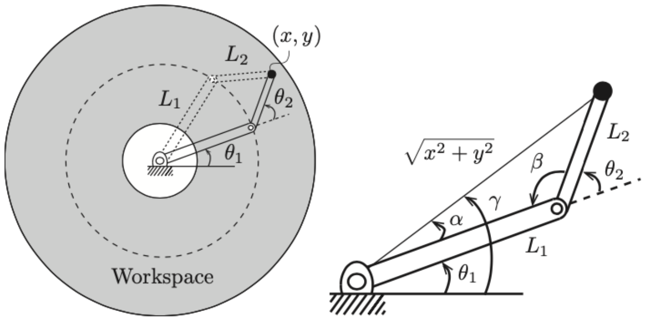
Figure 6.1: Inverse kinematics of a 2R planar open chain.

- For a general n DoF open chain with FK $T(\theta) \in SE(3)$, $\theta \in \Re^n$, the IK problem can be stated as:

  given a homogeneous transform $X \in SE(3)$, find solutions θ that satisfy $T(\theta) = X$.

  For example, the number of IK solutions will be zero, one, and two. When there are two solutions, they are called lefty and righty solutions, or elbow-up and elbow-down solutions.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} L_1 c_1 + L_2 c_{12} \\ L_1 s_1 + L_2 s_{12} \end{bmatrix}$$



(a) A workspace, and lefty and righty configurations.

(b) Geometric solution.

Figure 6.1: Inverse kinematics of a 2R planar open chain.

- Using the law of cosines, we have

$$L_1^2 + L_2^2 - 2L_1L_2\cos\beta = x^2 + y^2 \quad \rightarrow \quad \beta = \cos^{-1}\left(\frac{L_1^2 + L_2^2 - x^2 - y^2}{2L_1L_2}\right)$$

$$L_1^2 + x^2 + y^2 - 2L_1\sqrt{x^2+y^2}\cos\alpha = L_2^2 \quad \rightarrow \quad \alpha = \cos^{-1}\left(\frac{L_1^2 + x^2 + y^2 - L_2^2}{2L_1\sqrt{x^2+y^2}}\right)$$

- Using $\gamma = atan2(y, x) = \tan^{-1}\frac{y}{x}$ in the range $(-\pi, \pi]$, the righty solution becomes

$$\theta_1 = \gamma - \alpha \qquad \theta_2 = \pi - \beta$$

- The lefty solution is

$$\theta_1 = \gamma + \alpha \qquad \theta_2 = -\pi + \beta$$

- If $x^2 + y^2$ lies outside the range $[L_1 - L_2, L_1 + L_2]$, then no solution exists.

1 Analytic Inverse Kinematics

- Let us consider the FK of a spatial six-dof open chain in the following PoE form:

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M$$

- Given some end-effector frame $X \in SE(3)$, the IK problem is to find solutions

$$\theta \in \Re^6 \quad \text{satisfying} \quad T(\theta) = X$$

- As a typical example, we derive analytic inverse kinematic solutions for the PUMA and Stanford arms.

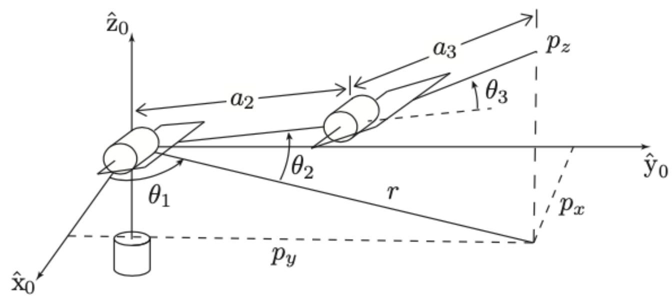


Figure 6.2: Inverse position kinematics of a 6R PUMA-type arm.

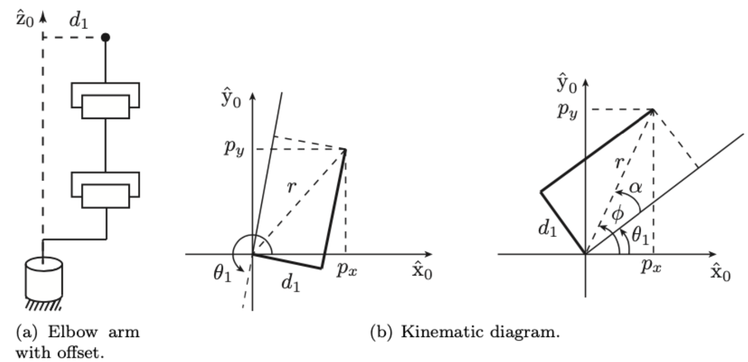


Figure 6.3: A 6R PUMA-type arm with a shoulder offset.

1.1 6R PUMA-Type Arm

When the arm is placed in its zero position:

1. the two shoulder joint axes intersect orthogonally at a common point, with joint axis 1 aligned in the z_0-direction and joint axis 2 aligned in the y_0-direction

2. joint axis 3 (the elbow joint) lies in the x_0-y_0-plane and is aligned parallel with joint axis 2

3. joint axes 4, 5, and 6 (the wrist joints) intersect orthogonally at a common point (the wrist center) to form an orthogonal wrist and, for the purposes of this example, we assume that these joint axes are aligned in the z_0-, y_0-, and x_0-directions, respectively.

4. The lengths of links 2 and 3 are a_2 and a_3, respectively.

5. The arm may also have an offset at the shoulder (right figure)

6. The inverse kinematics problem for PUMA-type arms can be decoupled into inverse-position and inverse-orientation subproblems

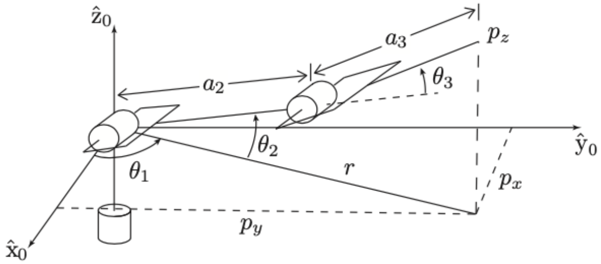


Figure 6.2: Inverse position kinematics of a 6R PUMA-type arm.

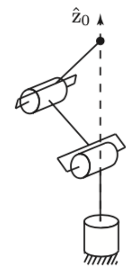


Figure 6.4: Singular configuration of the zero-offset 6R PUMA-type arm.

Zero-offset $d_1 = 0$

Consider the simple case of a zero-offset PUMA-type arm. Express all vectors in terms of fixed-frame coordinates, and denote the components of the wrist center $p \in \Re^3$ by $p = (p_x, p_y, p_z)$.

- Projecting p onto the $x_0 - y_0$-plane, it can be seen that

$$\theta_1 = atan2(p_y, p_x)$$

  In addition, we can get both θ_2 and θ_3 from (r, p_z) using the previous two-link manipulator kinematics.

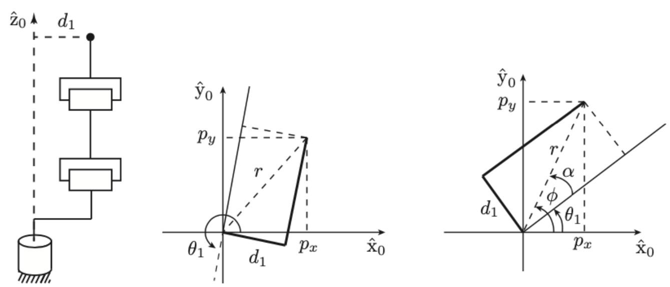
- Second solution for θ_1

$$\theta_1 = atan2(p_y, p_x) + \pi$$

  when the original solution for θ_2 is replaced by $\pi - \theta_2$.

- As long as $p_x, p_y \neq 0$, both these solutions are valid.

- When $p_x = p_y = 0$, the arm is in a singular configuration, and there are infinitely many possible solutions for θ_1. (right figure)



(a) Elbow arm with offset.

(b) Kinematic diagram.

Figure 6.3: A 6R PUMA-type arm with a shoulder offset.

If there is an offset $d_1 \neq 0$

- There will be two solutions for θ_1, the lefty and righty solutions

$$\theta_1 = \phi - \alpha \qquad\qquad \theta_1 = \pi + \phi + \alpha$$

where $\phi = atan2(p_y, p_x)$ and $\alpha = atan2\left(d_1, \sqrt{r^2 - d_1^2}\right)$ with $r^2 = p_x^2 + p_y^2$

- Determining angles θ_2 and θ_3 for the PUMA-type arm now reduces to the IK for a planar two-link chain:

$$\cos\theta_3 = \frac{r^2 - d_1^2 + p_z^2 - a_2^2 - a_3^2}{2a_2a_3} = D \quad \rightarrow \quad \theta_3 = atan2\left(\pm\sqrt{1 - D^2}, D\right)$$

Two solutions for θ_3 correspond to elbow-up and elbow-down configurations for two-link arm.

- θ_2 can be obtained in a similar fashion as

$$\theta_2 = atan2\left(p_z, \sqrt{r^2 - d_1^2}\right) - atan2\left(a_3s_3, a_2 + a_3c_3\right)$$

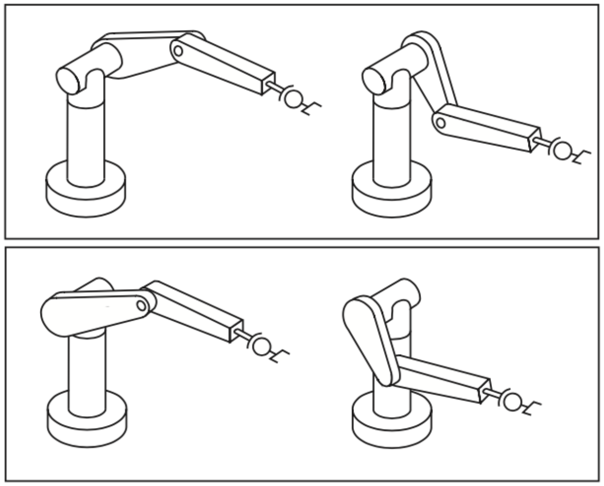


Figure 6.5: Four possible inverse kinematics solutions for the 6R PUMA-type arm with shoulder offset.

- PUMA-type arm with an offset will have four solutions to the inverse position problem.
- The postures in the upper panel are lefty solutions (elbow-up and elbow-down), while those in the lower panel are righty solutions (elbow-up and elbow-down).
- Let us solve the inverse orientation problem of finding $(\theta_4, \theta_5, \theta_6)$ given the end-effector orientation. This problem is completely straightforward:

$$\text{unknown} \quad e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} = e^{-[\mathcal{S}_1]\theta_1} e^{-[\mathcal{S}_2]\theta_2} e^{-[\mathcal{S}_3]\theta_3} X M^{-1} \quad \text{known}$$

- Since $\mathcal{S}_4 = (0,0,1,0,0,0)$, $\mathcal{S}_5 = (0,1,0,0,0,0)$, and $\mathcal{S}_6 = (1,0,0,0,0,0)$, the wrist joint angles $(\theta_4, \theta_5, \theta_6)$ can be determined as the solution to

$$Rot(\hat{z}, \theta_4) Rot(\hat{y}, \theta_5) Rot(\hat{x}, \theta_6) = R \quad \text{from} \quad e^{-[\mathcal{S}_1]\theta_1} e^{-[\mathcal{S}_2]\theta_2} e^{-[\mathcal{S}_3]\theta_3} X M^{-1}$$

which correspond exactly to the ZYX Euler angles, derived in Appendix B.

1.2 Stanford-Type Arms

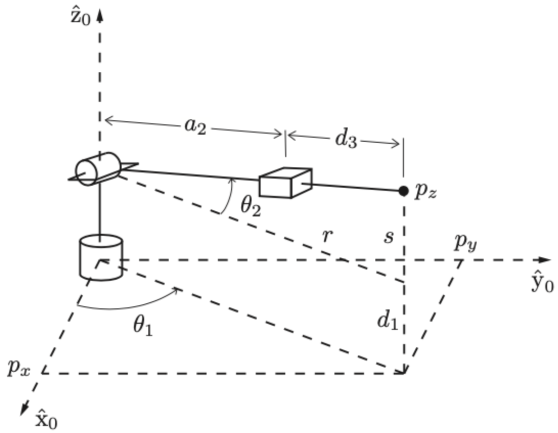


Figure 6.6: The first three joints of a Stanford-type arm.

- If the elbow joint in a 6R PUMA-type arm is replaced by a prismatic joint, we then have an RRPRRR Stanford-type arm.
- The first joint variable θ_1 can be found in similar fashion to the PUMA-type arm: $\theta_1 = atan2(p_y, p_x)$ (provided that p_x and p_y are not both zero).
- The variable θ_2 is then found to be $\theta_2 = atan2(s, r)$ with $s = p_z - d_1$ and $r^2 = p_x^2 + p_y^2$.
- Similarly to the case of the PUMA-type arm, a second solution for θ_1 and θ_2 is given by $\theta_1 = \pi + atan2(p_y, p_x)$ and $\theta_2 = \pi - atan2(s, r)$
- The translation distance θ_3 is found from the relation

$$(\theta_3 + a_2)^2 = r^2 + s^2 \qquad \rightarrow \qquad \theta_3 = \sqrt{p_x^2 + p_y^2 - (p_z - d_1)^2} - a_2$$