## 3.3 Exponential Coordinate Representation of Rigid-Body Motions

### Exponential Coordinates of Rigid-Body Motions

- Chasles-Mozzi theorem: every rigid-body displacement can be expressed as a displacement along a fixed screw axis $\mathcal{S}$ in space.

- (In the previous lecture) The exponential coordinates $\hat{\omega}\theta$ for rotation:

$$\begin{aligned} \text{matrix exponential :} \quad & [\hat{\omega}]\theta \in so(3) \quad \rightarrow \quad R = e^{[\hat{\omega}]\theta} \in SO(3) \\ \text{matrix logarithm :} \quad & R = e^{[\hat{\omega}]\theta} \in SO(3) \quad \rightarrow \quad [\hat{\omega}]\theta \in so(3) \end{aligned}$$

- By analogy to the exponential coordinates $\hat{\omega}\theta$ for rotations $R$, let us define the six-dimensional exponential coordinates of a homogeneous transformation $T$ as $\mathcal{S}\theta \in \Re^6$, where $\theta$ is the distance that must be traveled along the screw axis to take a frame from the origin $I$ to $T$.
  - If the pitch of the screw axis $\mathcal{S} = (\omega, v)$ is finite then $\|\omega\| = 1$ and $\theta$ corresponds to the angle of rotation about the screw axis.
  - If the pitch of the screw is infinite then $\omega = 0$ and $\|v\| = 1$ and $\theta$ corresponds to the linear distance traveled along the screw axis.

- Also by analogy to the rotation case, let us define a matrix exponential (exp) and matrix logarithm (log):

$$\begin{aligned} \text{matrix exponential :} \quad & [\mathcal{S}]\theta \in se(3) \quad \rightarrow \quad T = e^{[\mathcal{S}]\theta} \in SE(3) \\ \text{matrix logarithm :} \quad & T = e^{[\mathcal{S}]\theta} \in SE(3) \quad \rightarrow \quad [\mathcal{S}]\theta \in se(3) \end{aligned}$$

- Expanding the matrix exponential in series form leads to

$$e^{[\mathcal{S}]\theta} = I_{4\times4} + [\mathcal{S}]\theta + [\mathcal{S}]^2\frac{\theta^2}{2!} + [\mathcal{S}]^3\frac{\theta^3}{3!} + \cdots = \begin{bmatrix} I_{3\times3} + [\omega]\theta + [\omega]^2\frac{\theta^2}{2!} + [\omega]^3\frac{\theta^3}{3!} + \cdots & v\theta + [\omega]v\frac{\theta^2}{2!} + [\omega]^2 v\frac{\theta^3}{3!} + \cdots \\ 0_{3\times1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0_{3\times1} & 1 \end{bmatrix}$$

where

$$[\mathcal{S}]^2 = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\omega]^2 & [\omega]v \\ 0 & 0 \end{bmatrix} \qquad [\mathcal{S}]^3 = \begin{bmatrix} [\omega]^2 & [\omega]v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\omega]^3 & [\omega]^2 v \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} G(\theta) &= I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + [\omega]^3\frac{\theta^4}{4!} + [\omega]^4\frac{\theta^5}{5!} + \cdots \\ &= I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} - [\omega]\frac{\theta^4}{4!} - [\omega]^2\frac{\theta^5}{5!} + \cdots \\ &= I\theta + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right)[\omega] + \left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \cdots\right)[\omega]^2 \\ &= I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2 \end{aligned}$$

**Proposition 3.13.** *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ or if $\omega = 0$ and $\|v\| = 1$, then, for any distance $\theta \in \Re$ traveled along the axis, respectively,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & (I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v \\ 0_{3\times1} & 1 \end{bmatrix} \qquad e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0_{3\times1} & 1 \end{bmatrix}$$

**Matrix Logarithm of Rigid-Body Motions**

- On the contrary, for given an arbitrary $T = (R, p) \in SE(3)$, one can always find a screw axis $\mathcal{S} = (\omega, v)$ and a scalar $\theta$ such that

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0_{3\times 1} & 1 \end{bmatrix} \in SE(3) \qquad \rightarrow \qquad [\mathcal{S}]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0_{3\times 1} & 0 \end{bmatrix} \in se(3)$$

where $[\mathcal{S}]\theta$ is the matrix logarithm of $T = (R, p)$.

**Algorithm 3.2.** *Given $(R, p)$ written as $T \in SE(3)$, find a $\theta \in [0, \pi]$ and a screw axis $S = (\omega, v) \in \Re^6$ (where at least one of $\|\omega\| = 1$ and $\|v\|$ is unity) such that $e^{[\mathcal{S}]\theta} = T$. The vector $\mathcal{S}\theta \in \Re^6$ comprises the exponential coordinates for $T$ and the matrix $[\mathcal{S}]\theta \in se(3)$ is the matrix logarithm of $T$.*

- *If $R = I$ then set $\omega = 0$, $v = \frac{p}{\|p\|}$ and $\theta = \|p\|$*

- *Otherwise, use the matrix logarithm on $SO(3)$ to determine $\omega$ (written as in the $SO(3)$ algorithm) and $\theta$ for $R$. Then $v$ is calculated as*

$$v = G(\theta)^{-1} p$$

*where*

$$G(\theta)^{-1} = \frac{1}{\theta} I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2$$

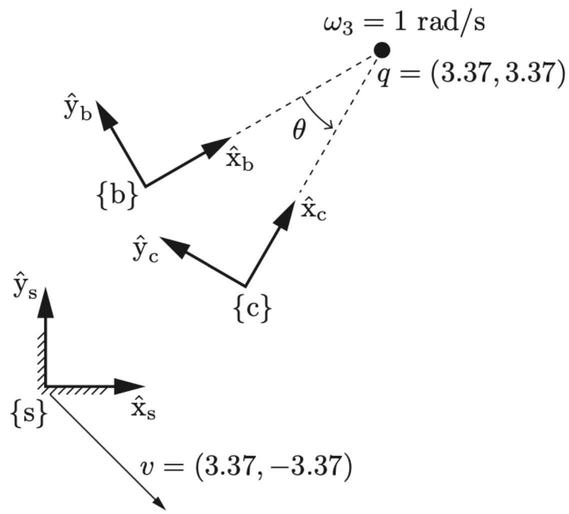


**Figure 3.20:** Two frames in a plane.

**Example 3.4.** *Assume the he rigid-body motion is confined to the $\hat{x}_s - \hat{y}_s$ plane. The initial frame $\{b\}$ and final frame $\{c\}$ can be represented by the $SE(3)$ matrices*

$$T_{sb} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 1 \\ \sin 30^\circ & \cos 30^\circ & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{sc} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 & 2 \\ \sin 60^\circ & \cos 60^\circ & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Assume the exponential coordinates are expressed in the fixed frame {s}, then we have

$$e^{[\mathcal{S}]\theta}T_{sb} = T_{sc} \qquad \rightarrow \qquad e^{[\mathcal{S}]\theta} = T_{sc}T_{sb}^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 2.134 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & -1.232 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = T(R,p)$$

- Find $\theta$ from the rotation matrix

$$\theta = \cos^{-1}\left(\frac{tr(R)-1}{2}\right) = \cos^{-1}\frac{\sqrt{3}}{2} = 30^\circ = 0.5236\ rad$$

- Find $\hat{\omega}$ from the rotation matrix

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^T) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} \qquad \rightarrow \qquad \hat{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Find $v$ from $v = G(\theta)^{-1}p$

$$v = G(\theta)^{-1}p = \left[\frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2\right]p$$

$$= \left[1.9096\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 0.5\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0.0439\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]\begin{bmatrix} 2.134 \\ -1.232 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} 1.8658 & 0.5 & 0 \\ -0.5 & 1.8658 & 0 \\ 0 & 0 & 1.9096 \end{bmatrix} \begin{bmatrix} 2.134 \\ -1.232 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.37 \\ -3.37 \\ 0 \end{bmatrix}$$

- **Find the screw axis $\mathcal{S}$, and the exponential coordinates (normalized twist)**

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3.37 \\ -3.37 \\ 0 \end{bmatrix} \qquad \mathcal{S}\theta = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.5236 \\ 1.7622 \\ -1.7622 \\ 0 \end{bmatrix}$$

- Alternatively,
  - **we can observe that the displacement is not a pure translation - $T_{sb}$ and $T_{sc}$ have rotation components that differ by an angle of $\theta = 30°$ and $\omega_z = 1$.**
  - **we can also graphically determine the point $q = (q_x, q_y)$ in the $\hat{x}_s - \hat{y}_s$ plane through which the screw axis passes; for our example this point is given by $q = (3.37, 3.37)$.**

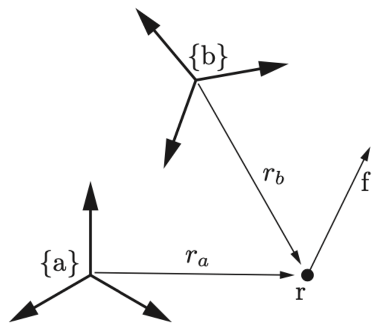


**Figure 3.21:** Relation between wrench representations $\mathcal{F}_a$ and $\mathcal{F}_b$.

# 4 Wrenches

- Consider a linear force f acting on a rigid body at a point r.
- Defining a reference frame {a}, the point r can be represented as $r_a \in \Re^3$ and the force f can be represented as $f_a \in \Re^3$.
- This force creates a torque or moment $m_a \in \Re^3$ in the {a} frame:

$$m_a = r_a \times f_a$$

- Just as with twists, we can merge the moment and force into a single six-dimensional spatial force, or wrench, expressed in the {a} frame,

$$\mathcal{F}_a = \begin{bmatrix} m_a \\ f_a \end{bmatrix} \in \Re^6$$

- A wrench with a zero linear component is called a pure moment.

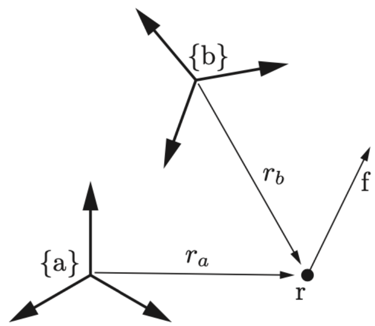


**Figure 3.21:** Relation between wrench representations $\mathcal{F}_a$ and $\mathcal{F}_b$.

- A wrench and twist in the {a} frame can be represented in another frame {b}
- Since the power is coordinate-independent quantity, we have

$$\mathcal{V}_b^T \mathcal{F}_b = \mathcal{V}_a^T \mathcal{F}_a = ([Ad_{T_{ab}}]\mathcal{V}_b)^T \mathcal{F}_a = \mathcal{V}_b^T [Ad_{T_{ab}}]^T \mathcal{F}_a$$

**Proposition 3.14.** *Given a wrench* F*, represented in* $\{a\}$ *as* $\mathcal{F}_a$ *and in* $\{b\}$ *as* $\mathcal{F}_b$*, two representations are related by*

$$\mathcal{F}_b = [Ad_{T_{ab}}]^T \mathcal{F}_a = Ad_{T_{ab}}^T \mathcal{F}_a$$
$$\mathcal{F}_a = [Ad_{T_{ba}}]^T \mathcal{F}_b = Ad_{T_{ba}}^T \mathcal{F}_b$$

- Spatial wrench $\mathcal{F}_s$ in {s} and body wrench $\mathcal{F}_b$ in {b}

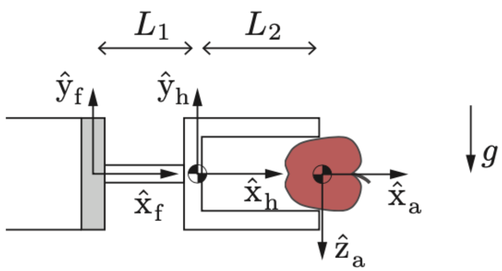


**Figure 3.22:** A robot hand holding an apple subject to gravity.

**Example 3.5.** *What is the force and torque measured by the six-axis force-torque sensor b/w the hand and the arm? (mass of apple :* $0.1[kg]$*, mass of hand :* $0.5[kg]$*,* $g = 10[m/s^2]$*,* $L_1 = 0.1[m]$*, and* $L_2 = 0.15[m]$*) where* $\{f\}$ *is attached to the force-torque sensor,* $\{h\}$ *to CoM of hand,* $\{a\}$ *to CoM of apple*

- Gravitational wrench on the hand in {h} is given by

$$\mathcal{F}_h = (0, 0, 0, 0, -5N, 0)$$

- Gravitational wrench on the apple in {a} is given by

$$\mathcal{F}_h = (0, 0, 0, 0, 0, 1N)$$

- Transformation matrices

$$T_{hf} = \begin{bmatrix} 1 & 0 & 0 & -0.1[m] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{af} = \begin{bmatrix} 1 & 0 & 0 & -0.25[m] \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Wrench measured by the six-axis force-torque sensor is

$$\mathcal{F}_f = [Ad_{T_{hf}}]^T \mathcal{F}_h + [Ad_{T_{af}}]^T \mathcal{F}_a$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 1 & 0 \\ 0 & -0.1 & 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -5 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.25 & 0 & -1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.5 \\ 0 \\ -5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -0.25 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.75 \\ 0 \\ -6 \\ 0 \end{bmatrix}$$

| Rotations | Rigid-Body Motions |
|---|---|
| $R \in SO(3) : 3 \times 3$ matrices<br>$R^{\mathrm{T}}R = I, \det R = 1$ | $T \in SE(3) : 4 \times 4$ matrices<br>$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$,<br>where $R \in SO(3), p \in \mathbb{R}^3$ |
| $R^{-1} = R^{\mathrm{T}}$ | $T^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$ |
| change of coordinate frame:<br>$R_{ab}R_{bc} = R_{ac}, \quad R_{ab}p_b = p_a$ | change of coordinate frame:<br>$T_{ab}T_{bc} = T_{ac}, \quad T_{ab}p_b = p_a$ |
| rotating a frame {b}:<br>$R = \mathrm{Rot}(\hat{\omega}, \theta)$<br>$R_{sb'} = RR_{sb}$:<br>rotate $\theta$ about $\hat{\omega}_s = \hat{\omega}$<br>$R_{sb''} = R_{sb}R$:<br>rotate $\theta$ about $\hat{\omega}_b = \hat{\omega}$ | displacing a frame {b}:<br>$T = \begin{bmatrix} \mathrm{Rot}(\hat{\omega}, \theta) & p \\ 0 & 1 \end{bmatrix}$<br>$T_{sb'} = TT_{sb}$: rotate $\theta$ about $\hat{\omega}_s = \hat{\omega}$<br>(moves {b} origin), translate $p$ in {s}<br>$T_{sb''} = T_{sb}T$: translate $p$ in {b},<br>rotate $\theta$ about $\hat{\omega}$ in new body frame |
| unit rotation axis is $\hat{\omega} \in \mathbb{R}^3$,<br>where $\\|\hat{\omega}\\| = 1$ | "unit" screw axis is $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$,<br>where either (i) $\\|\omega\\| = 1$ or<br>(ii) $\omega = 0$ and $\\|v\\| = 1$ |
| | for a screw axis $\{q, \hat{s}, h\}$ with finite $h$,<br>$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$ |
| angular velocity is $\omega = \hat{\omega}\dot{\theta}$ | twist is $\mathcal{V} = \mathcal{S}\dot{\theta}$ |

| Rotations (cont.) | Rigid-Body Motions (cont.) |
|---|---|
| for any 3-vector, e.g., $\omega \in \mathbb{R}^3$, $[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$ identities, $\omega, x \in \mathbb{R}^3, R \in SO(3)$: $[\omega] = -[\omega]^{\mathrm{T}}, [\omega]x = -[x]\omega$, $[\omega][x] = ([x][\omega])^{\mathrm{T}}, R[\omega]R^{\mathrm{T}} = [R\omega]$ | for $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, $[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$ (the pair $(\omega, v)$ can be a twist $\mathcal{V}$ or a "unit" screw axis $\mathcal{S}$, depending on the context) |
| $\dot{R}R^{-1} = [\omega_s], \quad R^{-1}\dot{R} = [\omega_b]$ | $\dot{T}T^{-1} = [\mathcal{V}_s], \quad T^{-1}\dot{T} = [\mathcal{V}_b]$ |
| | $[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$ identities: $[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}]$, $[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}] = [\mathrm{Ad}_{T_1T_2}]$ |
| change of coordinate frame: $\hat{\omega}_a = R_{ab}\hat{\omega}_b, \quad \omega_a = R_{ab}\omega_b$ | change of coordinate frame: $\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \quad \mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$ |
| exp coords for $R \in SO(3)$: $\hat{\omega}\theta \in \mathbb{R}^3$ | exp coords for $T \in SE(3)$: $\mathcal{S}\theta \in \mathbb{R}^6$ |
| $\exp : [\hat{\omega}]\theta \in so(3) \to R \in SO(3)$ $R = \mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} =$ $I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$ | $\exp : [\mathcal{S}]\theta \in se(3) \to T \in SE(3)$ $T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & * \\ 0 & 1 \end{bmatrix}$ where $* =$ $(I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$ |
| $\log : R \in SO(3) \to [\hat{\omega}]\theta \in so(3)$ algorithm in Section 3.2.3.3 | $\log : T \in SE(3) \to [\mathcal{S}]\theta \in se(3)$ algorithm in Section 3.3.3.2 |
| moment change of coord frame: $m_a = R_{ab}m_b$ | wrench change of coord frame: $\mathcal{F}_a = (m_a, f_a) = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{F}_b$ |

# 5 Homework : Chapter 3

- Please solve and submit Exercise 3.5, 3.6, 3.7, 3.14, 3.17, 3.18, 3.21, 3.22, 3.23, 3.24, 3.25, 3.39, till April 9 (upload it as a pdf form or email me)
- If you let me know what the numbers you cannot solve until April 7, I will include the solving process in the next lecture.