# 2 Rotations and Angular Velocities

## 2.1 Rotation Matrices

- Among nine entries in the rotation matrix $R$, only three can be chosen independently.
  1. The unit norm condition: $\hat{x}_b, \hat{y}_b, \hat{z}_b$ are all unit vectors, i.e.,

$$r_{11}^2 + r_{21}^2 + r_{31}^2 = 1,$$
$$r_{12}^2 + r_{22}^2 + r_{32}^2 = 1,$$
$$r_{13}^2 + r_{23}^2 + r_{33}^2 = 1$$

  2. The orthogonality condition: $\hat{x}_b \cdot \hat{y}_b = \hat{x}_b \cdot \hat{z}_b = \hat{y}_b \cdot \hat{z}_b = 0$

$$r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} = 0$$
$$r_{11}r_{13} + r_{21}r_{23} + r_{31}r_{33} = 0$$
$$r_{12}r_{13} + r_{22}r_{23} + r_{32}r_{33} = 0$$

- These six constraints can be expressed more compactly as a single set of constraints on $R$,

$$R^T R = I$$

- The frame is right-handed if $\hat{x}_b \times \hat{y}_b = \hat{z}_b$, and the left-handed if $\hat{x}_b \times \hat{y}_b = -\hat{z}_b$. Thus it can be obtained by using the determinant

$$\det R = 1 \quad \leftarrow \quad \det R = \hat{z}_b^T(\hat{x}_b \times \hat{y}_b) = \hat{z}_b^T\hat{z}_b = 1 \quad \textbf{right-handed}$$

$$\det R = -1 \quad \leftarrow \quad \det R = \hat{z}_b^T(\hat{x}_b \times \hat{y}_b) = -\hat{z}_b^T\hat{z}_b = -1 \quad \textbf{left-handed}$$

**Definition 3.1.** *The special orthogonal group $SO(3)$, also known as the group of rotation matrices, is the set of all $3 \times 3$ real matrices $R$ that satisfy*

1. $R^T R = I$
2. $\det R = 1$

**Definition 3.2.** *The special orthogonal group $SO(2)$ is the set of all $2 \times 2$ real matrices $R$ that satisfy*

1. $R^T R = I$
2. $\det R = 1$

From the definition it follows that every $R \in SO(2)$ can be written

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $\theta \in [0, 2\pi)$.

**Properties of Rotation Matrices**

- The sets of rotation matrices $SO(2)$ and $SO(3)$ are called groups because they satisfy the properties required of a mathematical group.
- Specifically, a group consists of a set of elements and an operation on two elements (matrix multiplication for $SO(n)$) such that, for all $A$, $B$ in the group, the following properties are satisfied:
  - closure: $AB$ is also in the group.
  - associativity: $(AB)C = A(BC)$.
  - identity element existence: There exists an element $I$ in the group.
  - inverse element existence: $\exists$ an element $A^{-1}$ in the group $\ni AA^{-1} = A^{-1}A = I$.
- More specifically, $SO(n)$ groups are also called matrix Lie groups (where "Lie" is pronounced "Lee") because the elements of the group form a differentiable manifold.

**Proposition 3.1.** *The inverse of a rotation matrix $R \in SO(3)$ is also a rotation matrix, and it is equal to the transpose of $R$, i.e., $R^{-1} = R^T$.*

**Proposition 3.2.** *The product of two rotation matrices is a rotation matrix.*

**Proposition 3.3.** *Multiplication of rotation matrices is associative, $(R_1R_2)R_3 = R_1(R_2R_3)$, but generally not commutative, $R_1R_2 \neq R_2R_1$.*

**Proposition 3.4.** *For any vector $x \in \Re^3$ and $R \in SO(3)$, the vector $y = Rx$ has the same length as $x$.*

## Uses of Rotation Matrices

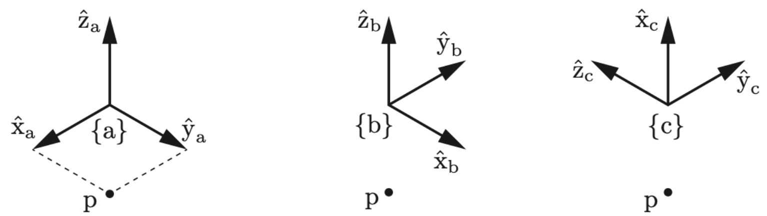


**Figure 3.7:** The same space and the same point p represented in three different frames with different orientations.

There are three major uses for a rotation matrix $R$:

1. to represent an orientation;
2. to change the reference frame in which a vector or a frame is represented; (operator)
3. to rotate a vector or a frame. (operator)

For a point $\mathrm{p}$ in the space, if a fixed space frame $\{\mathrm{s}\}$ is aligned with $\{\mathrm{a}\}$, then the orientations of the three frames relative to $\{\mathrm{s}\}$ and the location of the point $\mathrm{p}$ in these frames can be written

$$R_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \quad p_a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad p_b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad p_c = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

Note that $\{\mathrm{b}\}$ is obtained by rotating $\{\mathrm{a}\}$ about $\hat{z}_a$ by $90^\circ$, and $\{\mathrm{c}\}$ is obtained by rotating $\{\mathrm{b}\}$ about $\hat{y}_b$ by $-90^\circ$.

## Representing an orientation

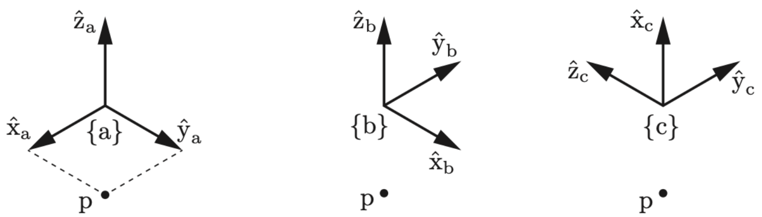


**Figure 3.7:** The same space and the same point p represented in three different frames with different orientations.

- When we write $R_c$, we are implicitly referring to the orientation of frame {c} relative to the fixed frame {s}.
- Its more explicit form is $R_{sc}$: we are representing the frame {c} of the second subscript relative to the frame {s} of the first subscript. For example, $R_{bc}$ is the orientation of {c} relative to {b}.
- If there is no possibility of confusion regarding the frames involved, we may simply write $R$.
- Inspecting Figure 3.7, we see that

$$R_{ac} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \qquad R_{ca} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

- A simple calculation shows that $R_{ac}R_{ca} = I$; that is, $R_{ac} = R_{ca}^{-1}$ or, equivalently, from Proposition 3.3, $R_{ac} = R_{ca}^T$.
- In fact, for any two frames {d} and {e},

$$R_{de} = R_{ed}^{-1} = R_{ed}^T.$$

### Changing the reference frame

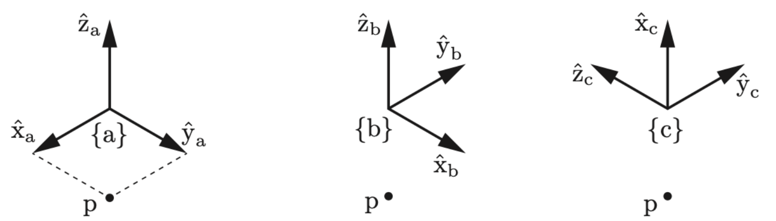


**Figure 3.7:** The same space and the same point p represented in three different frames with different orientations.

- The rotation matrix $R_{ab}$ represents the orientation of {b} in {a}, and $R_{bc}$ represents the orientation of {c} in {b}.
- A straightforward calculation shows that the orientation of {c} in {a} can be computed as

$$R_{ac} = R_{ab}R_{bc}$$

where $R_{ab}$ acts like an operator that changes the reference frame from {b} to {a} and $R_{bc}$ is a representation of the orientation.

$$R_{ac} = R_{ab}R_{bc} = \text{change reference frame from } \{b\} \text{ to } \{a\}\ (R_{bc}).$$

- Subscript cancellation rule

$$R_{ab}R_{bc} = R_{a\not{b}}R_{\not{b}c} = R_{ac}.$$

$$R_{ab}p_b = R_{a\not{b}}p_{\not{b}} = p_a$$

where the reference frame of a vector can be changed by a rotation matrix using the subscript cancellation rule.

## Rotating a vector or a frame

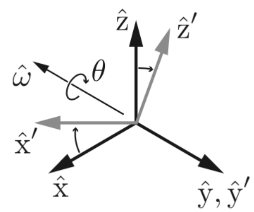


**Figure 3.8:** A coordinate frame with axes $\{\hat{x}, \hat{y}, \hat{z}\}$ is rotated by $\theta$ about a unit axis $\hat{\omega}$ (which is aligned with $-\hat{y}$ in this figure). The orientation of the final frame, with axes $\{\hat{x}', \hat{y}', \hat{z}'\}$, is written as $R$ relative to the original frame.

- Figure 3.8 shows a frame {c} initially aligned with {s} with axes $\{\hat{x}, \hat{y}, \hat{z}\}$.
- If we rotate the frame {c} about a unit axis $\hat{\omega}$ by an amount $\theta$, the new frame, {c'} has coordinate axes $\{\hat{x}', \hat{y}', \hat{z}'\}$. The rotation matrix $R = R_{sc'}$ represents the orientation of {c'} relative to {s}.
- Emphasizing our view of $R$ as a rotation operator, we can write for $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$

$$R = Rot(\hat{\omega}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix}$$

  where $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. Note that $Rot(\hat{\omega}, \theta) = Rot(-\hat{\omega}, -\theta)$.
- Typical examples of rotation operations about coordinate frame axes are

$$Rot(\hat{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad Rot(\hat{y}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad Rot(\hat{z}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

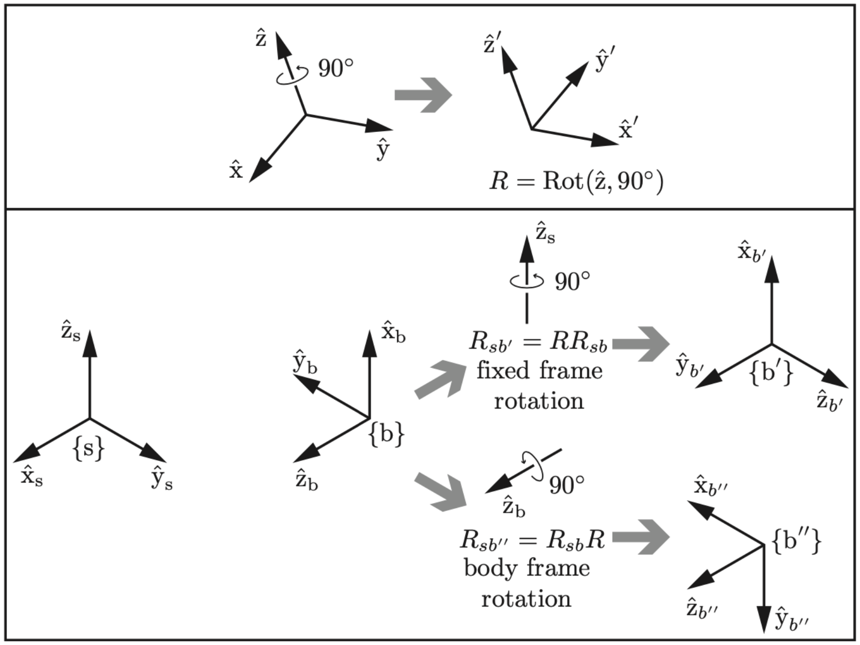


**Figure 3.9:** (Top) The rotation operator $R = \text{Rot}(\hat{z}, 90°)$ gives the orientation of the right-hand frame in the left-hand frame. (Bottom) On the left are shown a fixed frame {s} and a body frame {b}, which can be expressed as $R_{sb}$. The quantity $RR_{sb}$ rotates {b} by 90° about the fixed-frame axis $\hat{z}_s$ to {b′}. The quantity $R_{sb}R$ rotates {b} by 90° about the body-frame axis $\hat{z}_b$ to {b″}.

- To specify whether the axis of rotation is expressed in {s} or {b}, let us {b'} be the new frame after a rotation by $\theta$ about $\hat{\omega}_s = \hat{\omega}$ and {b"} be the new frame after a rotation by $\theta$ about $\hat{\omega}_b = \hat{\omega}$
- Representations of these new frames can be calculated as

$$R_{sb'} = \textbf{rotate by } R \textbf{ in \{s\} frame } (R_{sb}) = RR_{sb}$$

$$R_{sb''} = \textbf{rotate by } R \textbf{ in \{b\} frame } (R_{sb}) = R_{sb}R$$

- Premultiplying by $R = Rot(\hat{\omega}, \theta)$ yields a rotation about an axis $\hat{\omega}$ considered to be in the fixed frame, and postmultiplying by $R$ yields a rotation about $\hat{\omega}$ considered as being in the body frame.