# 제 3 장

## Rigid-Body Motions

- In the previous chapter, we have seen that a minimum of six numbers is needed to specify the position and orientation of a rigid body in three-dimensional physical space.
- In this chapter, we develop a systematic way to describe a rigid body's position and orientation which relies on attaching a reference frame to the body.
- The configuration of this frame w.r.t. a fixed reference frame is represented as a $4 \times 4$ matrix. $\rightarrow$ This matrix is an example of an implicit representation of the C-space.
- The actual six-dimensional space of rigid-body configuration is obtained by applying ten constraints to the 16-dimensional space of $4 \times 4$ real matices.
- For this purpose, this chapter has suggested
  - exponential coordinates (six-parameter representation of the configuration)
  - free vector (a geometric quantity with a length and a direction, but it is not rooted anywhere)
  - coordinate-free (when it does not have any coordinate frame)
  - spatial velocity or twist
  - spatial force or wrench

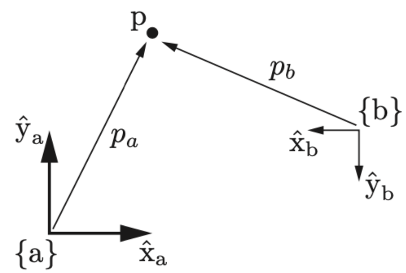


**Figure 3.1:** The point p exists in physical space, and it does not care how we represent it. If we fix a reference frame {a}, with unit coordinate axes $\hat{x}_a$ and $\hat{y}_a$, we can represent p as $p_a = (1, 2)$. If we fix a reference frame {b} at a different location, a different orientation, and a different length scale, we can represent p as $p_b = (4, -2)$.

- A coordinate-free point p in physical space can be represented as a vector $p \in \Re^n$ from the reference frame.
- A different choice of reference frame and length scale for physical space leads to a different representation $p \in \Re^n$ for the same point p in physical space, for example, $p_a$ in {a} reference frame and $p_b$ in {b} frame.
- Space frame, denoted {s}, has been defined as a fixed frame. For example, it might be attached to a corner of a room.
- Body frame, denoted {b}, is the stationary frame that is coincident with the moving body-attached frame at any instant. It may be chosen at the mass center of the moving rigid body.
- For simplicity, we will usually refer to a body frame {b} as a frame attached to a moving rigid body.

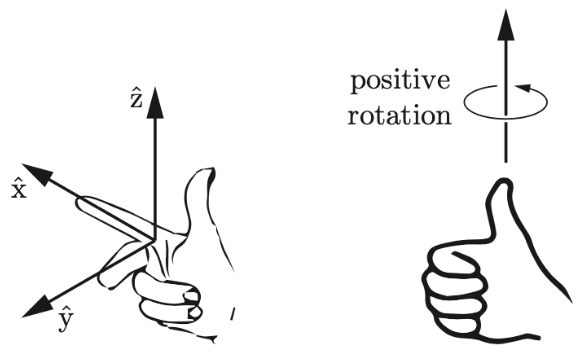


**Figure 3.2:** (Left) The x̂, ŷ, and ẑ axes of a right-handed reference frame are aligned with the index finger, middle finger, and thumb of the right hand, respectively. (Right) A positive rotation about an axis is in the direction in which the fingers of the right hand curl when the thumb is pointed along the axis.

- All reference frames are right-handed.
- If index finger is aligned with $\hat{x}$-axis and middle finger is aligned with $\hat{y}$-axis, then $\hat{z}$-axis is defined as thumb direction that the fingers of the right hand curl.

# 1 Rigid-Body Motions in the Plane

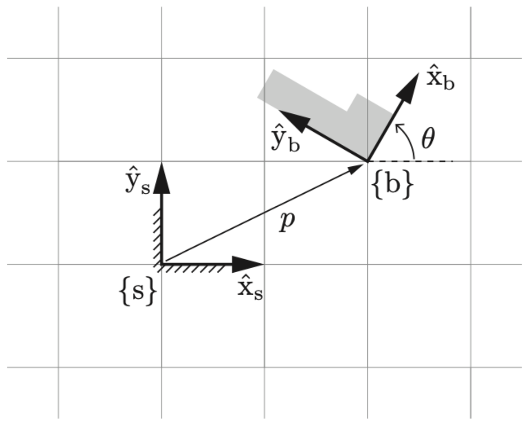


**Figure 3.3:** The body frame {b} is expressed in the fixed-frame coordinates {s} by the vector $p$ and the directions of the unit axes $\hat{x}_b$ and $\hat{y}_b$. In this example, $p = (2, 1)$ and $\theta = 60^\circ$, so $\hat{x}_b = (\cos\theta, \sin\theta) = (0.5, 1/\sqrt{2})$ and $\hat{y}_b = (-\sin\theta, \cos\theta) = (-1/\sqrt{2}, 0.5)$.

- Suppose that a length scale and a fixed reference frame {s} have been chosen with unit axes $\hat{x}_s$ and $\hat{y}_s$ as unit vectors.
- Similarly, we attach a reference frame with unit axes $\hat{x}_b$ and $\hat{y}_b$ to the planar body by using the body frame denoted {b} as a frame attached to a moving body.
- The body-frame origin p can be expressed in terms of the coordinate axes of {s} as

$$\begin{aligned} p &= p_x \hat{x}_s + p_y \hat{y}_s \\ &= 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned}$$

- The simplest way to describe the orientation of the body frame {b} relative to the fixed frame {s} is by specifying the angle $\theta$

$$\hat{x}_b = \cos\theta \hat{x}_s + \sin\theta \hat{x}_y = \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix} + \frac{\sqrt{3}}{2}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{bmatrix}$$

$$\hat{y}_b = -\sin\theta \hat{x}_s + \cos\theta \hat{x}_y = -\frac{\sqrt{3}}{2}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{3}}{2}\\\frac{1}{2}\end{bmatrix}$$

- Assuming we agree to express everything in terms of {s}, the point p can be represented as a column vector $p \in \Re^2$ of the form:

$$p = \begin{bmatrix}p_x\\p_y\end{bmatrix}$$

and two vectors $\hat{x}_b$ and $\hat{y}_b$ can also be written as column vectors and packaged into the following $2 \times 2$ rotation matrix $P$

$$P = \begin{bmatrix}\hat{x}_b & \hat{y}_b\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$$

- Although the rotation matrix $P$ consists of four numbers, they are subject to three constraints (each column of $P$ must be a unit vector, and the two columns must be orthogonal to each other), and the one remaining degree of freedom is parametrized by $\theta$.
- The pair $(P, p)$ provides a description of the orientation and position of {b} relative to {s}.

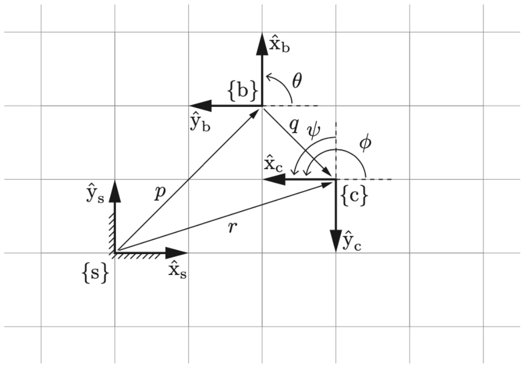


**Figure 3.4:** The frame {b} in {s} is given by $(P, p)$, and the frame {c} in {b} is given by $(Q, q)$. From these we can derive the frame {c} in {s}, described by $(R, r)$. The numerical values of the vectors $p$, $q$, and $r$ and the coordinate-axis directions of the three frames are evident from the grid of unit squares.

- Expressing {b} in {s} as the pair $(P, p)$, we have $p = \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

- Expressing {c} in {b} as the pair $(Q, q)$, $q = \begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $Q = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

- If we know $(Q, q)$ (the configuration of {c} relative to {b}) and $(P, p)$ (the configuration of {b} relative to {s}), we can compute the configuration of {c} relative to {s} as follows:

$$R = PQ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{convert } Q \text{ to the \{s\} frame}$$

$$r = Pq + p = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{convert } q \text{ to the \{s\} frame and vector-sum with } p$$

- Thus $(P, p)$ not only represents a configuration of {b} in {s}; it can also be used to convert the representation of a point or frame from {b} coordinates to {s} coordinates.

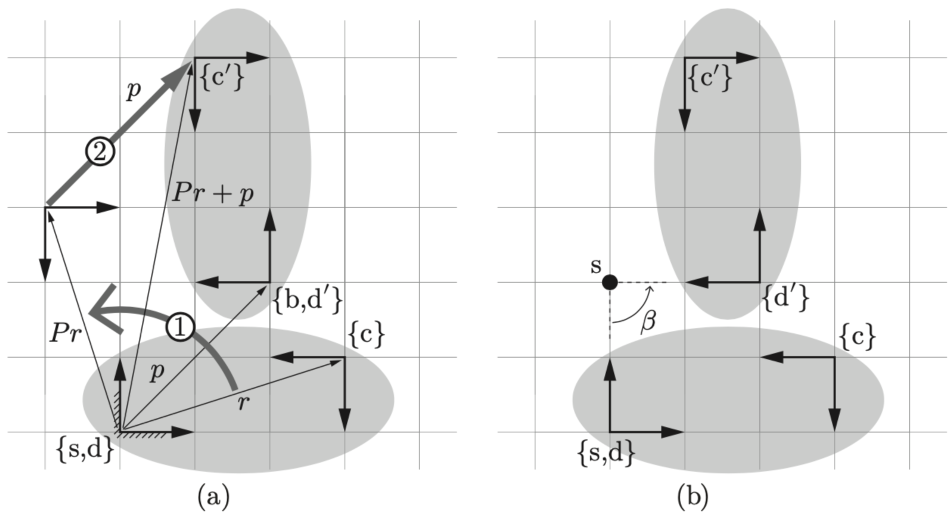


**Figure 3.5:** (a) The frame {d}, fixed to an elliptical rigid body and initially coincident with {s}, is displaced to {d′} (which is coincident with the stationary frame {b}), by first rotating according to $P$ then translating according to $p$, where $(P, p)$ is the representation of {b} in {s}. The same transformation takes the frame {c}, also attached to the rigid body, to {c′}. The transformation marked ① rigidly rotates {c} about the origin of {s}, and then transformation ② translates the frame by $p$ expressed in {s}. (b) Instead of viewing this displacement as a rotation followed by a translation, both rotation and translation can be performed simultaneously. The displacement can be viewed as a rotation of $\beta = 90°$ about a fixed point s.

- The rigid-body displacement (known as a rigid-body motion) is described by two sequential transformations, (ex) the rotation matrix-vector pair $(R, r)$ of {c} is moved to new frame $(R', r')$ of {c'}
  1. transformation rotates {c} according to $P$ : (ex) $R' = PR$
  2. transformation translates it by $p$ in {s} : (ex) $r' = Pr + p$
- A rotation matrix-vector pair $(P, p)$ can be used for three purpose:
  1. to represent a configuration of a rigid body in {s} (figure 3.3)
  2. to change the reference frame in which a vector or frame is represented (figure 3.4)
  3. to displace a vector or a frame (figure 3.5(a))

- Screw motion
  - Consider figure. 3.5(b), note that rigid-body motion, expressed as a rotation followed by a translation, can be obtained by simply rotating the body about a fixed point $s$ by an angle $\beta$.
  - This is a planar example of a screw motion.
  - Displacement can be parametrized by three screw coordinates $(\beta, s_x, s_y)$ in fixed frame {s}.
- Screw axis $\mathcal{S}$
  - Rotating about $s$ with a unit angular velocity $\omega = 1 rad/s$ means that a point at the origin of {s} frame moves at two units per second initially in the +$\hat{x}$-direction of the {s} frame, i.e., $v = (v_x, v_y) = (2, 0)$.
  - We can package these together in the three-vector $\mathcal{S} = (\omega, v_x, v_y) = (1, 2, 0)$, for a representation of the screw axis.
- Exponential coordinates $\mathcal{S}\theta$
  - Following this screw axis for an angle $\theta = \frac{\pi}{2}$ ($\beta = \frac{\pi}{2}$ in the figure) yields the final displacement.
  - Thus we can represent the displacement using the three coordinates $\mathcal{S}\theta = (\frac{\pi}{2}, \pi, 0)$.
  - These are called the exponential coordinates for the planar rigid-body displacement.
- Twist $\mathcal{V} = \mathcal{S}\dot{\theta}$
  - To represent the combination of an angular and a linear velocity, called a twist, we take a screw axis $\mathcal{S} = (\omega, v_x, v_y)$, where $\omega = 1$, and scale it by multiplying by some rotation speed, $\dot{\theta}$
  - The twist is $\mathcal{V} = S\dot{\theta}$
  - The net displacement obtained by rotating about the screw axis $\mathcal{S}$ by an angle $\theta$ is equivalent to the displacement obtained by rotating about $\mathcal{S}$ at a speed $\dot{\theta} = \theta$ for unit time, so $\mathcal{V} = \mathcal{S}\dot{\theta}$ can also be considered a set of exponential coordinates.

**Preview of the remainder of this chapter**

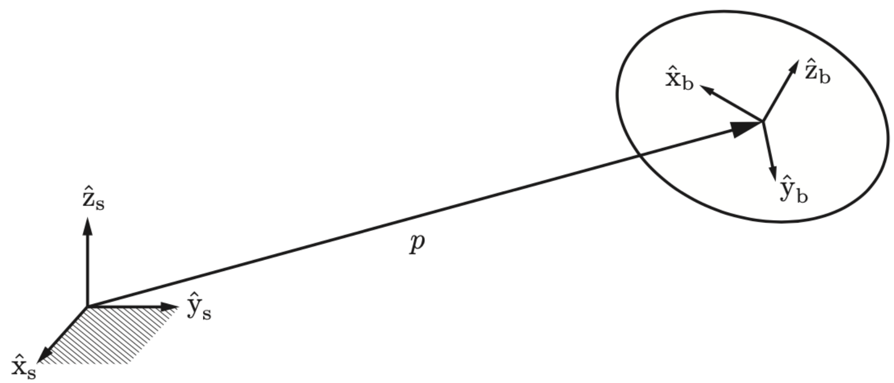


**Figure 3.6:** Mathematical description of position and orientation.

- Consider a rigid body occupying three-dimensional physical space, as shown in Figure 3.6.
- Assume that both the fixed frame {s} and body frame {b} have been chosen together with a length scale for physical space.
- All reference frames are right-handed - the unit axes $\{\hat{x}, \hat{y}, \hat{z}\}$ always satisfy $\hat{x} \times \hat{y} = \hat{z}$.
- In terms of the fixed-frame coordinates {s}, $\mathrm{p}$ can be expressed as

$$p = p_1\hat{x}_s + p_2\hat{y}_s + p_3\hat{z}_s$$

The axes of the body frame {b} can also be expressed as

$$\hat{x}_b = r_{11}\hat{x}_s + r_{21}\hat{y}_s + r_{31}\hat{z}_s$$
$$\hat{y}_b = r_{12}\hat{x}_s + r_{22}\hat{y}_s + r_{32}\hat{z}_s$$
$$\hat{z}_b = r_{13}\hat{x}_s + r_{23}\hat{y}_s + r_{33}\hat{z}_s$$

- Defining $p \in \Re^3$ and $R \in \Re^{3\times 3}$ as

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \qquad R = \begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- The 12 parameters given by $(R, p)$ then provide a description of the position and orientation of the rigid body relative to the fixed frame.
- Since the orientation of a rigid body has three degrees of freedom, only three of the nine entries in $R$ can be chosen independently.
- Every rigid-body displacement can be obtained by a finite rotation and translation about a fixed screw axis.