3 Newton-Euler Inverse Dynamics

- Consider the inverse dynamics (ID) problem for an n-link open chain connected by one-dof joints.
- Given the joint positions $\theta \in \Re^n$, velocities $\dot{\theta} \in \Re^n$, and accelerations $\ddot{\theta} \in \Re^n$, the objective is to calculate the right-hand side of the dynamics equation, ultimately to obtain τ

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$$

- Main result is a recursive ID algorithm consisting of a forward and a backward iteration stage.
  - In the forward iteration, the positions, velocities, and accelerations of each link are propagated from the base to the tip
  - In the backward iteration, the forces and moments experienced by each link are propagated from the tip to the base.

3.1 Derivation

- A body-fixed reference frame {i} is attached to the center of mass (CoM) of each link i, $i = 1, \cdots, n$.
- The base frame is denoted {0}, and a frame at the end-effector is denoted $\{n+1\}$, which is fixed in {n}.
- When the manipulator is at the home position, with all joint variables zero,

$$M_{i,j} \in SE(3) : \text{configuration of frame \{j\} in the frame \{i\}}$$
$$M_i = M_{0,i} : \text{configuration of \{i\} in the base frame \{0\}}$$

- With these definitions, $M_{i-1,i}$ and $M_{i,i-1}$ can be calculated as

$$M_{i-1,i} = M_{i-1}^{-1} M_i \quad \text{and} \quad M_{i,i-1} = M_i^{-1} M_{i-1}$$

- The screw axis for joint i, expressed in the link frame {i}, is $\mathcal{A}_i$. This same screw axis is expressed in the space (or base) frame {0} as $\mathcal{S}_i$, where the two are related by

$$\mathcal{A}_i = Ad_{M_i^{-1}}(\mathcal{S}_i)$$

- Defining $T_{i,j} \in SE(3)$ to be the configuration of frame {j} in {i} for arbitrary joint variables θ then $T_{i-1,i}(\theta_i)$, the configuration of {i} relative to $\{i-1\}$ given the joint variable θ_i, and $T_{i,i-1}(\theta_i) = T_{i-1,i}^{-1}(\theta_i)$ are calculated as

$$T_{i-1,i}(\theta_i) = M_{i-1,i} e^{[\mathcal{A}_i]\theta_i} \quad \text{and} \quad T_{i,i-1}(\theta_i) = e^{-[\mathcal{A}_i]\theta_i} M_{i,i-1}$$

- We further adopt the following notation:

  1. The twist of link frame {i}, expressed in frame-{i} coordinates, is denoted $\mathcal{V}_i = (\omega_i, v_i)$
  2. The wrench transmitted through joint i to link frame {i}, expressed in frame-{i} coordinates, is denoted $\mathcal{F}_i = (m_i, f_i)$.
  3. Let $\mathcal{G}_i \in \Re^{6\times 6}$ denote the spatial inertia matrix of link i, expressed relative to link frame {i}. Since we are assuming that all link frames are situated at the link CoM, $\mathcal{G}_i$ has the block-diagonal form

$$\mathcal{G}_i = \begin{bmatrix} \mathcal{I}_i & 0_{3\times 3} \\ 0_{3\times 3} & \mathrm{m}_i I \end{bmatrix}$$

  where $\mathcal{I}_i$ denotes the 3×3 rotational inertia matrix of link i and m_i is the link mass.

- With these definitions, we can recursively calculate the twist and acceleration of each link, moving from the base to the tip.

- The twist $\mathcal{V}_i$ of link i is the sum of the twist of link $i-1$, but expressed in {i}, and the added twist due to the joint rate $\dot{\theta}_i$:

$$\mathcal{V}_i = \mathcal{A}_i\dot{\theta}_i + [Ad_{T_{i,i-1}}]\mathcal{V}_{i-1}$$

- The accelerations $\dot{\mathcal{V}}_i$ can also be found recursively. Taking the time derivative, we get

$$\dot{\mathcal{V}}_i = \mathcal{A}_i\ddot{\theta}_i + [Ad_{T_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + \frac{d}{dt}([Ad_{T_{i,i-1}}])\mathcal{V}_{i-1}$$

- To calculate the final term in this equation, we express $T_{i,i-1}$ and $\mathcal{A}_i$ as

$$T_{i,i-1} = \begin{bmatrix} R_{i,i-1} & p \\ 0_{3\times 1} & 1 \end{bmatrix} \qquad \text{and} \qquad \mathcal{A}_i = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

Then

$$\begin{aligned}
\frac{d}{dt}([Ad_{T_{i,i-1}}])\mathcal{V}_{i-1} &= \frac{d}{dt}\left(\begin{bmatrix} R_{i,i-1} & 0_{3\times 3} \\ [p]R_{i,i-1} & R_{i,i-1} \end{bmatrix}\right)\mathcal{V}_{i-1} \\
&= \begin{bmatrix} -[\omega\dot{\theta}_i]R_{i,i-1} & 0_{3\times 3} \\ -[v\dot{\theta}_i]R_{i,i-1} - [\omega\dot{\theta}_i][p]R_{i,i-1} & -[\omega\dot{\theta}_i]R_{i,i-1} \end{bmatrix}\mathcal{V}_{i-1} \\
&= \begin{bmatrix} -[\omega\dot{\theta}_i] & 0_{3\times 3} \\ -[v\dot{\theta}_i] & -[\omega\dot{\theta}_i] \end{bmatrix}\begin{bmatrix} R_{i,i-1} & 0_{3\times 3} \\ [p]R_{i,i-1} & R_{i,i-1} \end{bmatrix}\mathcal{V}_{i-1} = -[ad_{\mathcal{A}_i\dot{\theta}_i}]\mathcal{V}_i = [ad_{\mathcal{V}_i}]\mathcal{A}_i\dot{\theta}_i
\end{aligned}$$

- Substituting this result into acceleration, we get

$$\dot{\mathcal{V}}_i = \mathcal{A}_i\ddot{\theta}_i + [Ad_{T_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + [ad_{\mathcal{V}_i}]\mathcal{A}_i\dot{\theta}_i$$

i.e., the acceleration of link i is the sum of three components: a component due to the joint acceleration $\ddot{\theta}_i$, a component due to the acceleration of link $i-1$ expressed in {i}, and a velocity-product component.

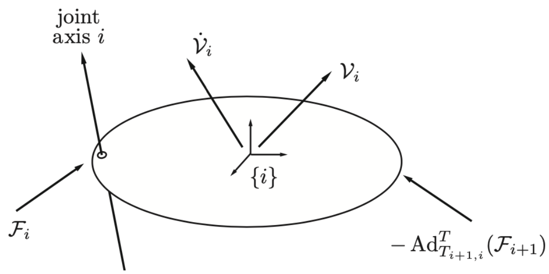

Figure 8.6: Free-body diagram illustrating the moments and forces exerted on link i.

- Once we have determined all the link twists and accelerations moving outward from the base, we can calculate the joint torques or forces by moving inward from the tip.
- The total wrench acting on link i is the sum of the wrench $\mathcal{F}_i$ transmitted through joint i and the wrench applied to the link through joint $i+1$ (or, for link n, the wrench applied to the link by the environment at the end-effector frame $\{n+1\}$), expressed in the frame i.

$$\mathcal{F}_b = \mathcal{G}_b\dot{\mathcal{V}}_b - [ad_{\mathcal{V}_b}]^T\mathcal{G}_b\mathcal{V}_b \qquad \rightarrow \qquad \mathcal{G}_i\dot{\mathcal{V}}_i - ad^T_{\mathcal{V}_i}(\mathcal{G}_i\mathcal{V}_i) = \mathcal{F}_i - Ad^T_{T_{i+1,i}}(\mathcal{F}_{i+1})$$

- Solving from the tip toward the base, at each link i we solve for the only unknown $\mathcal{F}_i$.
- Since joint i has only one-dof, five dimensions of the six-vector $\mathcal{F}_i$ are provided by the structure of the joint, and the actuator only has to provide the scalar force or torque in the direction of the joint's screw axis:

$$\tau_i = \mathcal{F}_i^T\mathcal{A}_i$$

where it provides the torques required at each joint, solving the ID problem.

3.2 Newton-Euler Inverse Dynamics (ID) Algorithm

- Initialization

  1. Attach a frame {0} to the base, frames {1} to {n} to the CoM of links {1} to {n}, and a frame $\{n+1\}$ at the end-effector, fixed in the frame {n}.
  2. Define $M_{i,i-1}$ to be the configuration of $\{i-1\}$ in {i} when $\theta_i = 0$.
  3. Let $\mathcal{A}_i$ be the screw axis of joint i expressed in {i}, and $\mathcal{G}_i$ be the 6×6 spatial inertia matrix of link i.
  4. Define $\mathcal{V}_0$ to be the twist of the base frame {0} expressed in {0} coordinates. (It is typically zero.)
  5. Let $g \in \Re^3$ be the gravity vector expressed in base-frame coordinates, and define $\dot{\mathcal{V}}_0 = (\dot{\omega}_0, \dot{v}_0) = (0, -g)$. (Gravity is treated as an acceleration of the base in the opposite direction.)
  6. Define $\mathcal{F}_{n+1} = \mathcal{F}_{tip} = (m_{tip}, f_{tip})$ to be the wrench applied to the environment by the end-effector, expressed in the end-effector frame $\{n+1\}$.

- Forward iterations : Given $\theta_i, \dot{\theta}_i, \ddot{\theta}_i$, for $i = 1$ to n do

$$T_{i,i-1}(\theta_i) = e^{-[\mathcal{A}_i]\theta_i} M_{i,i-1}$$

$$\mathcal{V}_i = Ad_{T_{i,i-1}}(\mathcal{V}_{i-1}) + \mathcal{A}_i \dot{\theta}_i$$

$$\dot{\mathcal{V}}_i = Ad_{T_{i,i-1}}(\dot{\mathcal{V}}_{i-1}) + ad_{\mathcal{V}_i}(\mathcal{A}_i)\dot{\theta}_i + \mathcal{A}_i \ddot{\theta}_i$$

- Backward iterations : Given $\mathcal{F}_{i+1}$, for $i = n$ to 1 do

$$\mathcal{F}_i = Ad^T_{T_{i+1,i}}(\mathcal{F}_{i+1}) + \mathcal{G}_i \dot{\mathcal{V}}_i - ad^T_{\mathcal{V}_i}(\mathcal{G}_i \mathcal{V}_i)$$

$$\tau_i = \mathcal{F}_i^T \mathcal{A}_i$$

4 Dynamic Equations in Closed Form

- The recursive ID algorithm is organized into a closed-form set of dynamics equations

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta)$$

- The sum of the kinetic energies of each link should be equal to $\frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta}$

$$\mathcal{K} = \frac{1}{2}\sum_{i=1}^{n} \mathcal{V}_i^T \mathcal{G}_i \mathcal{V}_i$$

  where $\mathcal{V}_i$ is the twist of link frame {i} and $\mathcal{G}_i$ is the spatial inertia matrix of link i (both are expressed in link-frame-{i} coordinates).

- Let $T_{0i}(\theta_1, \cdots, \theta_i)$ denote the forward kinematics from the base frame {0} to link frame {i}, and let $J_{ib}(\theta)$ denote the body Jacobian obtained from $T_{0i}^{-1}\dot{T}_{0i}$.

- Note that J_{ib} as defined is a $6 \times i$ matrix; we turn it into a $6 \times n$ matrix by filling in all entries of the last $n - i$ columns with zeros.

$$\mathcal{V}_i = J_{ib}(\theta)\dot{\theta}$$

- The kinetic energy can then be written

$$\mathcal{K} = \frac{1}{2}\dot{\theta}^T \left(\sum_{i=1}^{n} J_{ib}^T(\theta) \mathcal{G}_i J_{ib}(\theta) \right) \dot{\theta} \qquad \rightarrow \qquad M(\theta) = \sum_{i=1}^{n} J_{ib}^T(\theta) \mathcal{G}_i J_{ib}(\theta)$$

- Let us derive a closed-form set of dynamic equations by defining the following stacked vectors:

$$\mathcal{V} = \begin{bmatrix} \mathcal{V}_1 \\ \vdots \\ \mathcal{V}_n \end{bmatrix} \in \Re^{6n} \qquad \mathcal{F} = \begin{bmatrix} \mathcal{F}_1 \\ \vdots \\ \mathcal{F}_n \end{bmatrix} \in \Re^{6n}$$

- Further, define the following matrices:

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & 0_{6\times1} & \cdots & 0_{6\times1} \\ 0_{6\times1} & \mathcal{A}_2 & \cdots & 0_{6\times16} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{6\times1} & \cdots & \cdots & \mathcal{A}_n \end{bmatrix} \in \Re^{6n\times n} \qquad \mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & 0_{6\times6} & \cdots & 0_{6\times6} \\ 0_{6\times6} & \mathcal{G}_2 & \cdots & 0_{6\times6} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{6\times6} & \cdots & \cdots & \mathcal{G}_n \end{bmatrix} \in \Re^{6n\times6n}$$

$$[ad_{\mathcal{V}}] = \begin{bmatrix} [ad_{\mathcal{V}_1}] & 0_{6\times6} & \cdots & 0_{6\times6} \\ 0_{6\times6} & [ad_{\mathcal{V}_2}] & \cdots & 0_{6\times6} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{6\times6} & \cdots & \cdots & [ad_{\mathcal{V}_n}] \end{bmatrix} \in \Re^{6n\times6n} \qquad [ad_{\mathcal{A}\dot{\theta}}] = \begin{bmatrix} [ad_{\mathcal{A}_1\dot{\theta}_1}] & 0_{6\times6} & \cdots & 0_{6\times6} \\ 0_{6\times6} & [ad_{\mathcal{A}_2\dot{\theta}_2}] & \cdots & 0_{6\times6} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{6\times6} & \cdots & \cdots & [ad_{\mathcal{A}_n\dot{\theta}_n}] \end{bmatrix} \in \Re^{6n\times6n}$$

- We write $\mathcal{W}(\theta)$ to emphasize the dependence of $\mathcal{W}$ on θ.

$$\mathcal{W}(\theta) = \begin{bmatrix} 0_{6\times6} & 0_{6\times6} & \cdots & 0_{6\times6} & 0_{6\times6} \\ [Ad_{T_{21}}] & 0_{6\times6} & \cdots & 0_{6\times6} & 0_{6\times6} \\ 0_{6\times6} & [Ad_{T_{32}}] & \cdots & 0_{6\times6} & 0_{6\times6} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{6\times6} & 0_{6\times6} & \cdots & [Ad_{T_{n,n-1}}] & 0_{6\times6} \end{bmatrix} \in \Re^{6n\times6n}$$

- Finally, define the following stacked vectors:

$$\mathcal{V}_{base} = \begin{bmatrix} Ad_{T_{10}}(\mathcal{V}_0) \\ 0_{6\times1} \\ \vdots \\ 0_{6\times1} \end{bmatrix} \in \Re^{6n} \qquad \dot{\mathcal{V}}_{base} = \begin{bmatrix} Ad_{T_{10}}(\dot{\mathcal{V}}_0) \\ 0_{6\times1} \\ \vdots \\ 0_{6\times1} \end{bmatrix} \in \Re^{6n} \qquad \mathcal{F}_{tip} = \begin{bmatrix} 0_{6\times1} \\ \vdots \\ 0_{6\times1} \\ Ad^T_{T_{n+1,n}}(\mathcal{F}_{n+1}) \end{bmatrix} \in \Re^{6n}$$

Note that $\mathcal{A} \in \Re^{6n\times n}$ and $\mathcal{G} \in \Re^{6n\times6n}$ are constant block-diagonal matrices.

- With the above definitions, our earlier recursive inverse dynamics algorithm can be assembled into the following set of matrix equations:

$$\begin{aligned}
\mathcal{V} &= \mathcal{W}(\theta)\mathcal{V} + \mathcal{A}\dot{\theta} + \mathcal{V}_{base} \\
\dot{\mathcal{V}} &= \mathcal{W}(\theta)\dot{\mathcal{V}} + \mathcal{A}\ddot{\theta} - [ad_{\mathcal{A}\dot{\theta}}](\mathcal{W}(\theta)\mathcal{V} + \mathcal{V}_{base}) + \dot{\mathcal{V}}_{base} \\
\mathcal{F} &= \mathcal{W}(\theta)^T\mathcal{F} + \mathcal{G}\dot{\mathcal{V}} - [ad_{\mathcal{V}}]^T\mathcal{G}\mathcal{V} + \mathcal{F}_{tip} \\
\tau &= \mathcal{A}^T\mathcal{F}
\end{aligned}$$

- The matrix $\mathcal{W}(\theta)$ has the property that $\mathcal{W}^n(\theta) = 0_{6n\times 6n}$ (such a matrix is said to be nilpotent of order n), and one consequence verifiable through direct calculation is that

$$\begin{aligned}
(I_{6n\times 6n} - \mathcal{W})^{-1} &= I_{6n\times 6n} + \mathcal{W} + \mathcal{W}^2 + \cdots + \mathcal{W}^{n-1} = \mathcal{L}(\theta) \\
&= \begin{bmatrix} I_{6\times 6} & 0_{6\times 6} & 0_{6\times 6} & \cdots & 0_{6\times 6} \\ [Ad_{T_{21}}] & I_{6\times 6} & 0_{6\times 6} & \cdots & 0_{6\times 6} \\ [Ad_{T_{31}}] & [Ad_{T_{32}}] & I_{6\times 6} & \cdots & 0_{6\times 6} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ [Ad_{T_{n1}}] & [Ad_{T_{n2}}] & [Ad_{T_{n,3}}] & \cdots & I_{6\times 6} \end{bmatrix} \in \Re^{6n\times 6n}
\end{aligned}$$

- The earlier matrix equations can now be reorganized as follows:

$$
\begin{aligned}
\mathcal{V} &= \mathcal{L}(\theta)(\mathcal{A}\dot{\theta} + \mathcal{V}_{base}) \\
\dot{\mathcal{V}} &= \mathcal{L}(\theta)(\mathcal{A}\ddot{\theta} - [ad_{\mathcal{A}\dot{\theta}}](\mathcal{W}(\theta)\mathcal{V} + \mathcal{V}_{base}) + \dot{\mathcal{V}}_{base}) \\
\mathcal{F} &= \mathcal{L}(\theta)^T(\mathcal{G}\dot{\mathcal{V}} - [ad_{\mathcal{V}}]^T\mathcal{G}\mathcal{V} + \mathcal{F}_{tip}) \\
\tau &= \mathcal{A}^T\mathcal{F}
\end{aligned}
$$

- If the robot applies an external wrench $\mathcal{F}_{tip}$ at the end-effector, this can be included into the dynamics equation

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta) + J^T(\theta)\mathcal{F}_{tip}$$

where $J(\theta)$ denotes the Jacobian of the FK expressed in the same reference frame as $\mathcal{F}_{tip}$, and

$$
\begin{aligned}
M(\theta) &= \mathcal{A}^T\mathcal{L}^T(\theta)\mathcal{G}\mathcal{L}(\theta)\mathcal{A} \\
c(\theta, \dot{\theta}) &= -\mathcal{A}^T\mathcal{L}^T(\mathcal{G}\mathcal{L}(\theta)[ad_{\mathcal{A}\dot{\theta}}]\mathcal{W}(\theta) + [ad_{\mathcal{V}}]^T\mathcal{G})\mathcal{L}(\theta)\mathcal{A}\dot{\theta} \\
g(\theta) &= \mathcal{A}^T\mathcal{L}^T(\theta)\mathcal{G}\mathcal{L}(\theta)\dot{\mathcal{V}}_{base}
\end{aligned}
$$

5 Forward Dynamics of Open Chain

- The forward dynamics (FD) problem involves solving

$$M(\theta)\ddot{\theta} = \tau(t) - h(\theta, \dot{\theta}) - J^T(\theta)\mathcal{F}_{tip}$$

  for $\ddot{\theta}$, given $\theta, \dot{\theta}, \tau$ and the wrench $\mathcal{F}_{tip}$ applied by the end-effector (if applicable).

- Term $h(\theta, \dot{\theta})$ can be computed by calling the ID algorithm with $\ddot{\theta} = 0$ and $\mathcal{F}_{tip} = 0$.

- The inertia matrix $M(\theta)$ can be computed by n calling of the inverse dynamics algorithm to build $M(\theta)$ column by column.

  1. In each of the n calls, set $g = 0$, $\dot{\theta} = 0$, and $\mathcal{F}_{tip} = 0$.
  2. In the first call, the column vector $\ddot{\theta}$ is all zeros except for a 1 in the first row.
  3. In the second call, $\ddot{\theta}$ is all zeros except for a 1 in the second row, and so on.
  4. The τ vector returned by the ith call is the ith column of $M(\theta)$, and after n calls the $n \times n$ matrix $M(\theta)$ is constructed.

- With $M(\theta), h(\theta, \dot{\theta})$, and $\mathcal{F}_{tip}$, we can use any efficient algorithm for solving the equation of the form $M(\theta)\ddot{\theta} = b$, for $\ddot{\theta}$.

- The FD can be used to simulate the motion of the robot given its initial state, the joint forces-torques $\tau(t)$, and an optional external wrench $\mathcal{F}_{tip}(t)$, for $t \in [0, t_f]$.
- First define the function ForwardDynamics returning the solution:

$$\ddot{\theta} = FD(\theta, \dot{\theta}, \tau, \mathcal{F}_{tip})$$

- Defining the variables $q_1 = \theta, q_2 = \dot{\theta}$, the second-order dynamics can be converted to two first-order differential equations,

$$\dot{q}_1 = q_2$$
$$\dot{q}_2 = FD(\theta, \dot{\theta}, \tau, \mathcal{F}_{tip})$$

- The Euler integration of the robot dynamics is used

$$q_1(t + \delta t) = q_1(t) + q_2(t)\delta t$$
$$q_2(t + \delta t) = q_2(t) + FD(\theta, \dot{\theta}, \tau, \mathcal{F}_{tip})\delta t.$$

  Given a set of initial values for $q_1(0) = \theta(0)$ and $q_2(0) = \dot{\theta}(0)$, the above equations can be iterated forward in time to obtain the motion $\theta(t) = q_1(t)$ numerically.

- Euler Integration Algorithm for FD

  1. Inputs: The initial conditions $\theta(0)$ and $\dot{\theta}(0)$, the input torques $\tau(t)$ and wrenches at the end-effector $\mathcal{F}_{tip}(t)$ for $t \in [0, t_f]$, and the number of integration steps N.
  2. Initialization: Set the timestep $\delta t = \frac{t_f}{N}$, and set $\theta[0] = \theta(0), \dot{\theta}[0] = \dot{\theta}(0)$
  3. Iteration: For $k = 0$ to $N - 1$ do

$$\begin{aligned}\ddot{\theta}[k] &= FD(\theta[k], \dot{\theta}[k], \tau(k\delta t), \mathcal{F}_{tip}(k\delta)) \\ \theta[k+1] &= \theta[k] + \dot{\theta}[k]\delta t \\ \dot{\theta}[k+1] &= \dot{\theta}[k] + \ddot{\theta}[k]\delta t\end{aligned}$$

  4. Output: The joint trajectory $\theta(k\delta) = \theta[k]$, $\dot{\theta}(k\delta t) = \dot{\theta}[k]$, for $k = 0, \cdots, N$.

- The result of the numerical integration converges to the theoretical result as the number of integration steps N goes to infinity.

- Higher-order numerical integration schemes, such as fourth-order Runge-Kutta, can yield a closer approximation with fewer computations than the simple first-order Euler method.

6 Dynamics in the Task Space

- The dynamic equations change under a transformation to coordinates of the end-effector frame (task-space coordinates).
- Consider a six-dof open chain with joint space dynamics

$$\tau = M(\theta)\ddot{\theta} + h(\theta, \dot{\theta}) \qquad \theta \in \Re^6 \qquad \tau \in \Re^6$$

- The twist $\mathcal{V} = (\omega, v)$ of the end-effector is related to the joint velocity $\dot{\theta}$ by

$$\mathcal{V} = J(\theta)\dot{\theta}$$

  where $\mathcal{V}$ and $J(\theta)$ are always expressed in terms of the same reference frame.
- The time derivative $\dot{\mathcal{V}}$ is then

$$\dot{\mathcal{V}} = \dot{J}(\theta)\dot{\theta} + J(\theta)\ddot{\theta}$$

- At configurations θ where $J(\theta)$ is invertible, we have

$$\dot{\theta} = J^{-1}\mathcal{V} \qquad\qquad \ddot{\theta} = J^{-1}[\dot{\mathcal{V}} - \dot{J}J^{-1}\mathcal{V}]$$

- Substituting for $\dot{\theta}$ and $\ddot{\theta}$ leads to

$$\tau = M(\theta)[J^{-1}\dot{\mathcal{V}} - J^{-1}\dot{J}J^{-1}\mathcal{V}] + h(\theta, J^{-1}\mathcal{V})$$

- Pre-multiply both sides by J^{-T} to get

$$J^{-T}\tau = J^{-T}MJ^{-1}\dot{\mathcal{V}} - J^{-T}MJ^{-1}\dot{J}J^{-1}\mathcal{V} + J^{-T}h(\theta, J^{-1}\mathcal{V})$$

- Expressing $J^{-T}\tau$ as the wrench $\mathcal{F}$, the above can be written

$$\mathcal{F} = \Lambda(\theta)\dot{\mathcal{V}} + \eta(\theta, \mathcal{V})$$

  where

$$\Lambda(\theta) = J^{-T}MJ^{-1} \qquad \eta(\theta, \mathcal{V}) = J^{-T}h(\theta, J^{-1}\mathcal{V}) - \Lambda(\theta)\dot{J}J^{-1}\mathcal{V}$$

  These are the dynamic equations expressed in end-effector frame coordinates.

- If an external wrench $\mathcal{F}$ is applied to the end-effector frame then, assuming the actuators provide zero forces and torques, the motion of the end-effector frame is governed by these equations.

- Note that $J(\theta)$ must be invertible (i.e., there must be a one-to-one mapping between joint velocities and end-effector twists) in order to derive the task space dynamics above.

7 Homework : Chapter 8

- Please solve and submit Exercise 8.1, 8.2, 8.4, 8.6, 8.7 (upload it as a pdf form or email me)