3 Configuration Space: Topology and Representation

3.1 Configuration Space Topology

Robot's C-space - its dimension, its number of dof, and its shape (topology)

- Two spaces are topologically equivalent if one can be continuously deformed into the other w/o cutting or gluing
  - (sphere and football) A sphere can be deformed into a football simply by stretching w/o cutting or gluing, so these two spaces are topologically equivalent.
  - (sphere and plane) We cannot turn a sphere into a plane w/o cutting it, so a sphere and a plane are not topologically equivalent.
- Topologically distinct one-dimensional spaces includes the circle, the line, and a closed interval of the line.
  - The circle is written mathematically as S or S^1, indicating a one-dimensional sphere
  - The line as E or E^1, indicating a one-dimensional Euclidean (or flat) space. $\rightarrow$ since a point in E^1 is usually represented by a real number, it is often written as R or R^1 instead.
  - The closed-interval of the line, which contains its endpoints, can be written as $[a, b] \subset R^1$.
- In higher dimensions, R^n is the n-dimensional Euclidean space and S^n is the n-dimensional surface of a sphere in $(n+1)$-dimensional space.
- Note that the topology of a sphere is a fundamental property of the space itself and is independent of how we choose coordinates to represent points in the space .

- An open-interval of the line (a, b) does not include the endpoints a and b and is topologically equivalent to a line, since a line does not contain endpoints.

- A closed-interval of the line $[a, b]$ is not topologically equivalent to a line, since a line does not contain endpoints.

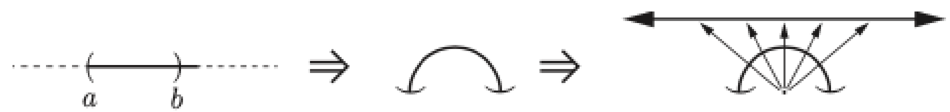

Figure 2.9: An open interval of the real line, denoted (a, b), can be deformed to an open semicircle. This open semicircle can then be deformed to the real line by the mapping illustrated: beginning from a point at the center of the semicircle, draw a ray that intersects the semicircle and then a line above the semicircle. These rays show that every point of the semicircle can be stretched to exactly one point on the line, and vice versa. Thus an open interval can be continuously deformed to a line, so an open interval and a line are topologically equivalent.

- Some C-space can be expressed as the Cartesian product of two or more spaces of lower dimension.

- In other words, points in such a C-space can be represented as the union of the representations of points in the lower-dimensional spaces.

  $\rightarrow$ For example, the C-space of a rigid body in the plane can be written as $R^2 \times S^1$, since the configuration can be represented as the concatenation of the coordinates (x, y) representing R^2 and an angle θ representing S^1.

1. C-space of a point on a plane can be written as R^2.

2. C-space of a spherical pendulum can be written as S^2.

3. C-space of a 2R robot arm can be written $S^1 \times S^1 = T^2$, where T^n is the n-dimensional surface of a torus in an $(n+1)$-dimensional space.

   Note that a sphere S^2 is not topologically equivalent to a torus T^2.

4. C-space of a PR joint can be written $R^1 \times S^1$.

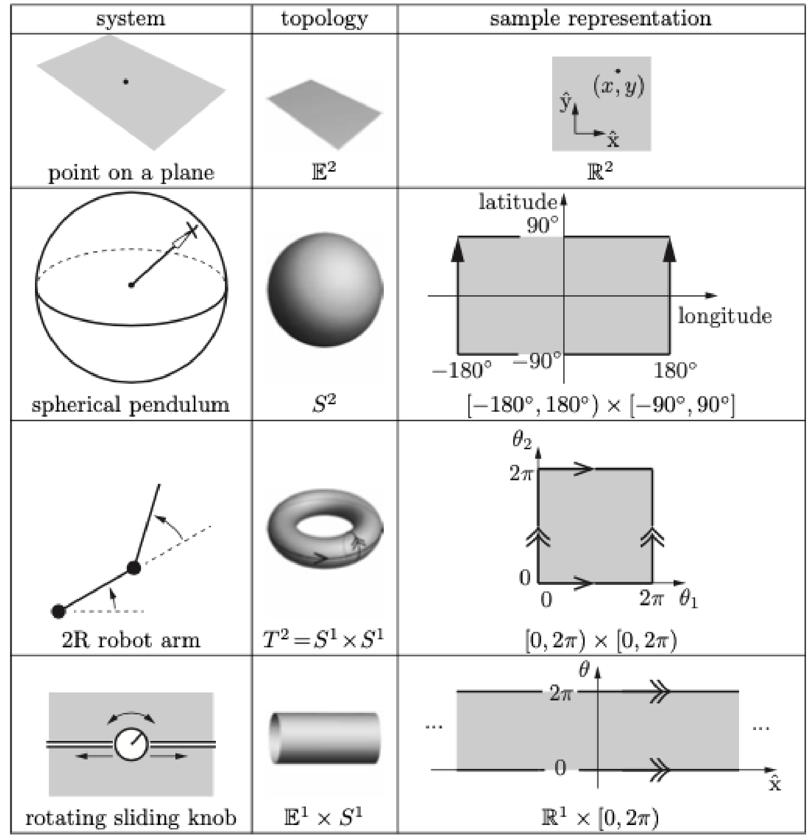

Table 2.2: Four topologically different two-dimensional C-spaces and example coordinate representations. In the latitude-longitude representation of the sphere, the latitudes −90° and 90° each correspond to a single point (the South Pole and the North Pole, respectively), and the longitude parameter wraps around at 180° and −180°; the edges with the arrows are glued together. Similarly, the coordinate representations of the torus and cylinder wrap around at the edges marked with corresponding arrows.

- The C-space of a planar rigid body (e.g., the chassis of a mobile robot) with a 2R robot arm can be written as $R^2 \times S^1 \times T^2 = R^2 \times T^3$.

- The C-space of a rigid body in three-dimensional space can be described by a point in R^3, plus a point on a two-dimensional sphere S^2, plus a point on a one-dimensional circle S^1, giving a total C-space of $R^3 \times S^2 \times S^1$.

3.2 Configuration Space Representation

- To perform computations, we must have a numerical representation of the space, consisting of a set of real numbers. - e.g., vector (an ordered list of variables)
- It is important to keep in mind that the representation of a space involves a choice, and therefore it is not as fundamental as the topology of the space, which is independent of the representation.
- Explicit parameterization of the space : consider the surface on the sphere with latitude-longitude coordinates
  - it is unsatisfactory if you are walking near the North Pole (latitude = $90°$) or South Pole (latitude = $-90°$), where taking a very small step can result in a large change in the coordinates.
  - The North and South Poles are singularities of the representation, and the existence of singularities is a result of the fact that a sphere does not have the same topology as a plane.
- To resolve singularities
  - use more than one coordinate chart on the space : As the configuration approaches a singularity in one chart, you simply switch to another chart where the North and South Poles are far from singularities. e.g., cylindrical coordinates
  - use an implicit representation of the space : We can use (x, y, z) in three-dimensional space with one constraint $x^2 + y^2 + z^2 = r^2$, since the a point moving smoothly around the sphere is represented by a smoothly changing (x, y, z), even at North and South Poles.
- Rotation matrix uses nine numbers, subject to six constraints, to represent the three orientation freedoms of a rigid body in space. - it is singularity-free and able to apply linear algebra.

4 Configuration and Velocity Constraints

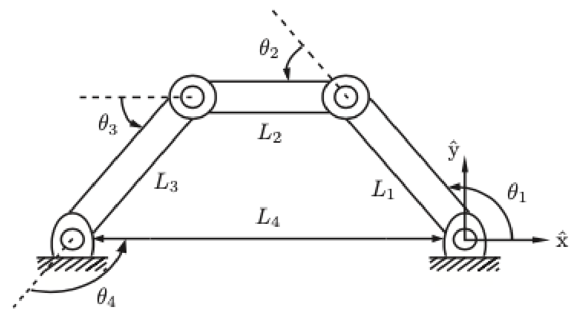

Figure 2.10: The four-bar linkage.

- For robots containing one or more closed loops, usually an implicit representation is more easily obtained than an explicit parametrization.
- Consider planar four-bar linkage

$$\text{x-coordinate} \quad g_1: \quad L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) + L_4\cos(\theta_1+\theta_2+\theta_3+\theta_4) = 0$$

$$\text{y-coordinate} \quad g_2: \quad L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3) + L_4\sin(\theta_1+\theta_2+\theta_3+\theta_4) = 0$$

$$\text{orientation} \quad g_3: \quad \theta_1+\theta_2+\theta_3+\theta_4 - 2\pi = 0$$

- It has one dof because of four variables $\theta_1, \theta_2, \theta_3, \theta_4$ and above three constraints (loop-closure equations)
- The set of all solutions forms a one-dimensional curve in the four-dimensional joint space and constitutes the C-space.

- The C-space can be implicitly represented by the column vector $\theta = [\theta_1, \cdots, \theta_n]^T \in R^n$ and the loop-closure equations of the form (constraints)

$$g(\theta) = \begin{bmatrix} g_1(\theta_1, \cdots, \theta_n) \\ \vdots \\ g_k(\theta_1, \cdots, \theta_n) \end{bmatrix} \in R^k$$

- Such constraints are known as holonomic constraints that reduce the dimension of the C-space.
- The C-space can be viewed as a surface of dimension $n - k$ (assuming that all constraints are independent) embedded in R^n.
- Let us take time derivative of the constraints as follow:

$$\begin{bmatrix} \frac{\partial g_1}{\partial \theta_1}\dot{\theta}_1 + \cdots + \frac{\partial g_1}{\partial \theta_1}\dot{\theta}_n \\ \vdots \\ \frac{\partial g_k}{\partial \theta_1}\dot{\theta}_1 + \cdots + \frac{\partial g_k}{\partial \theta_1}\dot{\theta}_n \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & \cdots & \frac{\partial g_1}{\partial \theta_n} \\ & \vdots & \\ \frac{\partial g_k}{\partial \theta_1} & \cdots & \frac{\partial g_k}{\partial \theta_n} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Above can be rewritten as the compact form:

$$\frac{\partial g}{\partial \theta}(\theta)\dot{\theta} = 0 \qquad \rightarrow \qquad A(\theta)\dot{\theta} = 0$$

- Velocity constraints of this form are called Pfaffian constraints. Inversely,
  - holonomic if Pfaffian constraints can be integrated to give equivalent configuration constraints
  - nonholonomic if Pfaffian constraints cannot be integrated

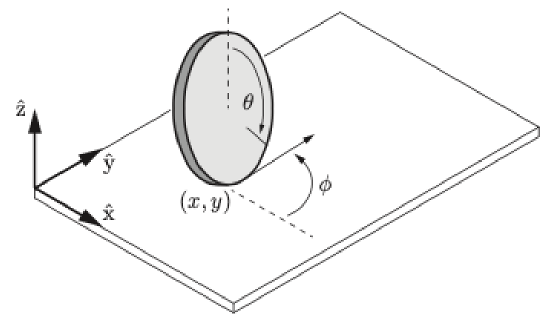

Figure 2.11: A coin rolling on a plane without slipping.

Consider an upright coin of radius r rolling on a plane.

- The configuration of the coin is given by the contact point (x, y) on the plane, the steering angle ϕ, and the angle of rotation θ.
- The C-space of the coin is therefore $R^2 \times T^2$, where T^2 is the two-dimensional torus.
- This C-space is four dimensional.
- Assuming that coin rolls w/o slipping, the coin must always rolls in the direction indicated by $(\cos\phi, \sin\phi)$ with forward speed $r\dot{\theta}$:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = r\dot{\theta} \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

- Collecting the four C-space coordinates into a single vector $q = [q_1, q_2, q_3, q_4]^T = [x, y, \phi, \theta]^T \in R^2 \times T^2$, the above no-slip rolling constraint can then be expressed in the form of Pfaffian constraints:

$$\begin{bmatrix} 1 & 0 & 0 & -r\cos q_3 \\ 0 & 1 & 0 & -r\sin q_3 \end{bmatrix} \dot{q} = 0 \qquad \rightarrow \qquad A(q)\dot{q} = 0$$

- Let us check whether the Pfaffian contstraints are integrable or not.

$$
\begin{aligned}
&\frac{\partial g_1}{\partial q_1} = 1 \quad \rightarrow \quad g_1(q) = q_1 + h_1(q_2, q_3, q_4) \\
&\frac{\partial g_1}{\partial q_2} = \frac{\partial h_1}{\partial q_2} = 0 \quad \rightarrow \quad h_1(q_2, q_3, q_4) = l_1(q_3, q_4) \quad \rightarrow \quad g_1(q) = q_1 + l_1(q_3, q_4) \\
&\frac{\partial g_1}{\partial q_3} = \frac{\partial l_1}{\partial q_3} = 0 \quad \rightarrow \quad l_1(q_3, q_4) = k(q_4) \quad \rightarrow \quad g_1(q) = q_1 + k_1(q_4) \\
&\frac{\partial g_1}{\partial q_4} = \frac{\partial k_1}{\partial q_4} = -r\cos q_3 \quad \rightarrow \quad k_1(q_4) = -rq_4 \cos q_3 \quad k_1 \text{ is a function of } q_3
\end{aligned}
$$

- No such g_1 exists and g_2 does not exist. $\rightarrow$ it is non-integrable.$\rightarrow$ nonholonomic
- Such Pfaffian constraints reduce the dimension of the feasible velocities of the system but do not reduce the dimension of the reachable C-space.
- Examples of nonholonomic constraints : conservation of momentum, rolling w/o slipping.

5 Task Space and Workspace

The task space is a space in which the robot's task can be naturally expressed.

- If the robot's task is to plot with a pen on a piece of paper, the task space would be R^2.
- The decision of how to define the task space is driven by the task, independently of the robot.

The workspace is a specification of the configurations that the end-effector of the robot can reach.

- Definition of the workspace is primarily driven by the robot's structure, independently of the task.

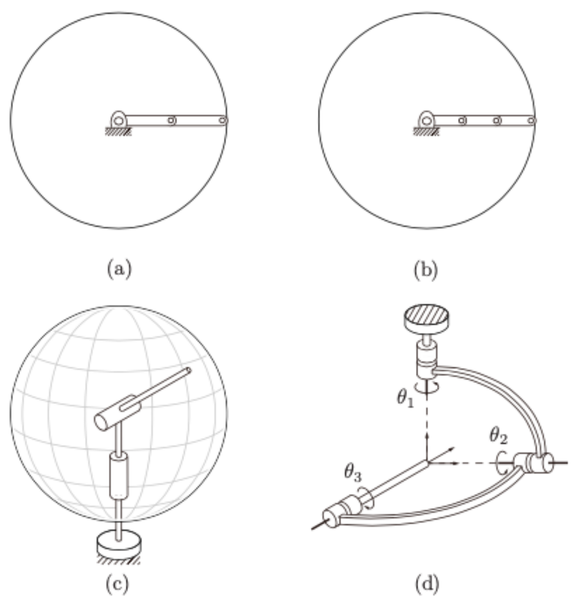

Figure 2.12: Examples of workspaces for various robots: (a) a planar 2R open chain; (b) a planar 3R open chain; (c) a spherical 2R open chain; (d) a 3R orienting mechanism.

1. C-Space of planar 2R : T^2, its workspace : R^2 (planar disk)
2. C-Space of planar 3R : T^3, its workspace : R^2 (planar disk)
3. C-Space of spherical 2R : S^2, its workspace : S^2 (surface of the sphere)
4. C-Space of wrist mechanism : $S^2 \times S^1$, its workspace : $S^2 \times S^1$ (all the orientation)

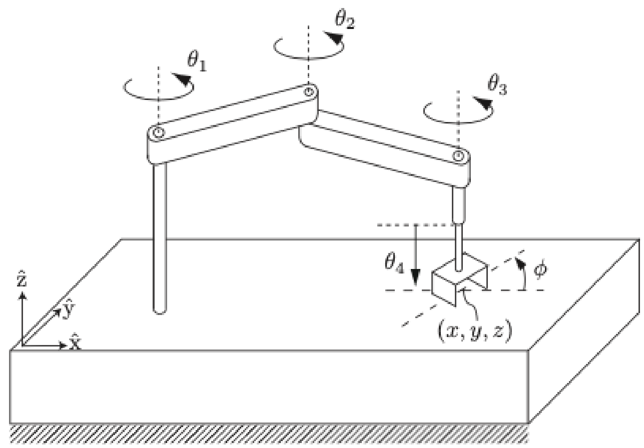

Figure 2.13: SCARA robot.

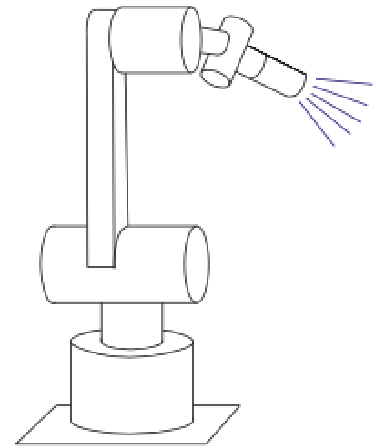

Figure 2.14: A spray-painting robot.

Example 2.7. *(SCARA robot)*

- *it has an RRRP open chain that is widely used for tabletop pick-and-place tasks.*
- *end-effector configuration is completely described by* (x, y, z, ϕ)
- *its task space would typically be defined as* $R^3 \times S^1$
- *its workspace would typically be defined as the reachable points in* $R^3 \times S^1$

Example 2.8. *(standard 6R industrial manipulator)*

- *assume that it is adapted to spray-painting applications*
- *end-effector configuration is completely described by* $(x, y, z, \phi, \theta, \psi)$
- *its task space would typically be defined as* $R^3 \times S^2$ *because the rotation about the nozzle axis do not matter*
- *its workspace would typically be defined as the reachable points in* $R^3 \times S^2$

6 Homework : Chapter 2

- Please solve and submit Exercise 2.5, 2.6, 2.7, 2.11, 2.12, 2.14, 2.19, 2.25, 2.26, 2.31, 2.32 till March 26 (upload it as a pdf form or email me)
- If you let me know what the numbers you cannot solve until March 23, I will include the solving process in the next lecture.