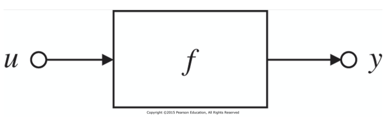
4 Equivalent Gain Analysis using Frequency Response: Describing Functions

- The *describing method*, based on the assumption that the input to the nonlinearity is sinusoidal, can be used to predict the behavior of a class of nonlinear systems.
- For a certain class of nonlinearities, it is possible to replace the nonlinearity by a *frequency-dependent equivalent gain* for analysis purposes.



- If the input signal $u(t)$ is *sinusoidal* of amplitude a,

$$u(t) = a \sin \omega t$$

then the output $y(t)$ will be *periodic* with a fundamental period equal to that of the input and consequently with a *Fourier series* described by

$$y(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t) = a_0 + \sum_{n=1}^{\infty} Y_i \sin(n\omega t + \theta_n)$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} y(t) \cos(n\omega t) d(\omega t) \qquad b_n = \frac{2}{\pi} \int_0^{\pi} y(t) \sin(n\omega t) d(\omega t) \qquad Y_n = \sqrt{a_n^2 + b_n^2} \qquad \theta_n = \tan^{-1} \frac{a_n}{b_n}$$

- Nonlinear element can be described by the *first fundamental components of this series* as if it were a linear system with a gain of Y_1 and phase of θ_1.

- The *describing function* is defined as the (complex) quantity that is a ratio of the amplitude of the fundamental component of the output of the nonlinear element to the amplitude of the sinusoidal input signal and is essentially an *equivalent frequency response* function:

$$\begin{aligned} DK &= K_{eq}(a, \omega) \\ &= \frac{a_1 j + b_1}{a} \\ &= \frac{Y_1(a, \omega)}{a} e^{j\theta_1} \\ &= \frac{Y_1(a, \omega)}{a} \angle \theta_1 \\ &= \frac{\sqrt{a_1^2 + b_1^2}}{a} \angle \tan^{-1} \frac{a_1}{b_1} \end{aligned}$$

  Hence, the *describing function is defined only on the $j\omega$ axis.*

- In the case of memoryless nonlinearities that are also an *odd* function, then the *coefficients of the Fourier series cosine terms are all zeros* and the describing function is simply

$$DF = K_{eq}(a) = \frac{b_1}{a}$$

  and is *independent* of the frequency ω. This is the usual case in control, and *saturation, relay, and dead-zone nonlinearities* all result in such describing functions.

- (Example 9.10) The *saturation* function is defined by

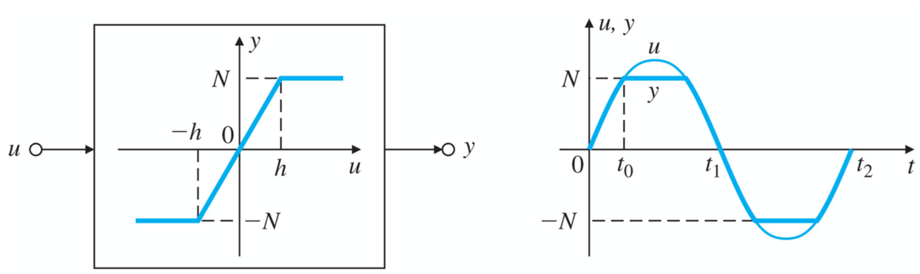
$$sat(x) = \begin{cases} +1 & x > 1 \\ x & |x| \le 1 \\ -1 & x < -1 \end{cases}$$

if the *slope* of the linear region is k and the final saturated values are $\pm N$, then the function is

$$y = N sat\left(\frac{k}{N}x\right) = N sat\left(\frac{x}{h}\right) \qquad \text{where} \quad k = \frac{N}{h}$$

Find the describing function for this nonlinearity?

1. For an input sinusoid of $u = a\sin\omega t$ with amplitude $a \le \frac{N}{k}$ (namely $a \le h$), the output is such that the DF is just a *gain of unity*.
2. With $a \ge \frac{N}{k}$ (namely, $a \ge h$), we need to compute the *amplitude and phase* of the fundamental component of the output.



(a) (b)

3. Since the saturation is an *odd* function, all the cosine terms are zeros and $a_0 = 0$.

4. Let us calculate b_1 as follow: $\frac{k}{N}a\sin(\omega t) = 1 \quad \rightleftarrows \quad \omega t_s = \sin^{-1}\frac{N}{ak}$.

$$
\begin{aligned}
b_1 &= \frac{2}{\pi}\int_0^{\pi} u(t)\sin(\omega t)d(\omega t) \\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} u(t)\sin(\omega t)d(\omega t) \\
&= \frac{4N}{\pi}\int_0^{\frac{\pi}{2}} sat\left(\frac{ak}{N}\sin(\omega t)\right)\sin(\omega t)d(\omega t) \\
&= \frac{4N}{\pi}\int_0^{\omega t_s} \frac{ak}{N}\sin^2(\omega t)d(\omega t) + \frac{4N}{\pi}\int_{\omega t_s}^{\frac{\pi}{2}} \sin(\omega t)d(\omega t) \\
&= \frac{4ak}{\pi}\int_0^{\omega t_s} \sin^2(\omega t)d(\omega t) + \frac{4N}{\pi}\int_{\omega t_s}^{\frac{\pi}{2}} \sin(\omega t)d(\omega t) \\
&= \frac{4ak}{\pi}\int_0^{\omega t_s} \frac{1-\cos(2\omega t)}{2}d(\omega t) + \frac{4N}{\pi}\int_{\omega t_s}^{\frac{\pi}{2}} \sin(\omega t)d(\omega t) \\
&= \frac{4ak}{\pi}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_0^{\omega t_s} - \frac{4N}{\pi}\cos\theta|_{\omega t_s}^{\frac{\pi}{2}} \\
&= \frac{4ak}{\pi}\left(\frac{\omega t_s}{2} - \frac{\sin 2\omega t_s}{4}\right) - \frac{4N}{\pi}(0 - \cos\omega t_s) \\
&= \frac{4ak}{\pi}\left(\frac{\omega t_s}{2} - \frac{\sin 2\omega t_s}{4}\right) + \frac{4N}{\pi}\cos\omega t_s \\
&= \frac{2ak}{\pi}\sin^{-1}\frac{N}{ak} - \frac{2N}{\pi}\sqrt{1 - \frac{N^2}{a^2k^2}} + \frac{4N}{\pi}\sqrt{1 - \frac{N^2}{a^2k^2}} \\
&= \frac{2ak}{\pi}\sin^{-1}\frac{N}{ak} + \frac{2N}{\pi}\sqrt{1 - \frac{N^2}{a^2k^2}} \\
&= \frac{2}{\pi}\left(ak\sin^{-1}\frac{N}{ak} + N\sqrt{1 - \frac{N^2}{a^2k^2}}\right)
\end{aligned}
$$

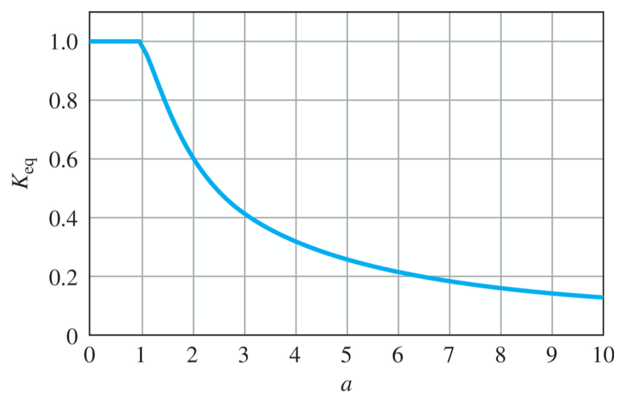
where the followings are used:

$$\sin \omega t_s = \frac{N}{ak} \qquad \cos \omega t_s = \sqrt{1 - \frac{N^2}{a^2k^2}} \qquad \sin 2\omega t_s = \frac{2N}{ak}\sqrt{1 - \frac{N^2}{a^2k^2}}$$

5. We finally obtain

$$K_{eq}(a) = \begin{cases} \frac{2}{\pi}\left(k \sin^{-1}\frac{N}{ak} + \frac{N}{a}\sqrt{1 - \frac{N^2}{a^2k^2}}\right) & \frac{ak}{N} > 1 \\ k & \frac{ak}{N} \leq 1 \end{cases}$$

6. A plot of $K_{eq}(a)$ indicates that it is a real function independent of frequency and results in no phase shifts.



7. It is seen that the describing function is *initially a constant and then decays* essentially as a function of the reciprocal of the input signal amplitude a.

- (Example 9.11) The *relay or sgn* function is defined by

$$sgn(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

then the function is

$$y = N sgn\,(x)$$

Find the describing function for this nonlinearity?

1. For an input sinusoid of $u = a \sin \omega t$ with amplitude $a \leq N$, the output is such that the DF is just a *gain of unity*.
2. With $a \geq N$, we need to compute the *amplitude and phase* of the fundamental component of the output.
3. Since the sign is an *odd* function, all the cosine terms are zeros and $a_0 = 0$.

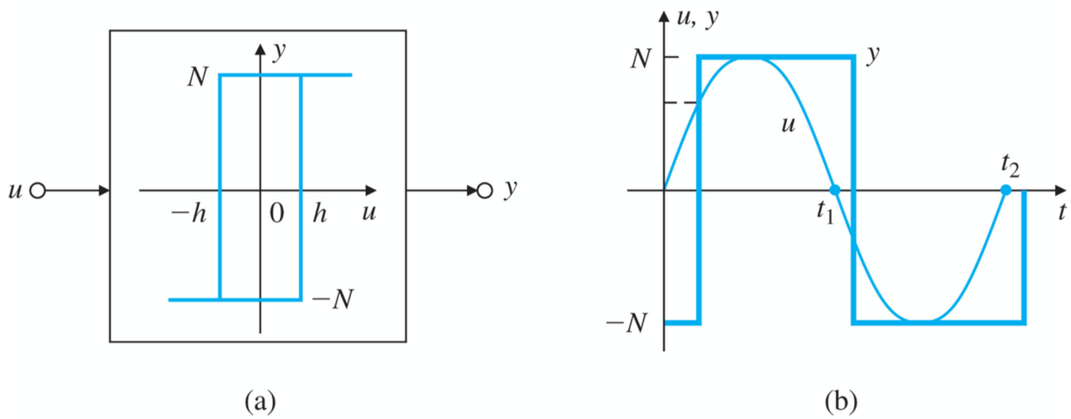
4. **Let us calculate b_1 as follow:** $a\sin(\omega t) = N \quad \rightleftarrows \quad \omega t_s = \sin^{-1}\frac{N}{a}$.

$$
\begin{aligned}
b_1 &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} u(t)\sin(\omega t)d(\omega t) \\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} N sgn\left(a\sin(\omega t)\right)\sin(\omega t)d(\omega t) \\
&= \frac{4N}{\pi}\int_0^{\frac{\pi}{2}} \sin(\omega t)d(\omega t) \\
&= \frac{4N}{\pi}(-\cos\theta)_0^{\frac{\pi}{2}} \\
&= \frac{4N}{\pi}(0-(-1)) \\
&= \frac{4N}{\pi}
\end{aligned}
$$

5. **We finally obtain**

$$K_{eq}(a) = \frac{4N}{a\pi}$$

- Next we consider a *nonlinearity with memory* such as magnetic recording device and backlash.



- (Example 9.12) Consider the *relay function with hysteresis* shown in Fig. 9.30(a). Find the describing function for this nonlinearity

  1. Assume $u = a \sin \omega t$. The output is a square wave with amplitude N as long as the input amplitude a is greater than the hysteresis level h.
  2. From Fig. 9.30(b), it is seen that the square wave lags the input in time. The lag time can be calculated as the time when

$$a \sin \omega t = h \qquad \rightarrow \qquad \omega_s t = \sin^{-1} \frac{h}{a}$$

3. Let us calculate a_1 as follow:

$$\begin{aligned}
a_1 &= \frac{2}{\pi}\int_0^{\pi} u(t)\cos(\omega t)d(\omega t) \\
&= \frac{2}{\pi}\int_0^{\omega_s t} u(t)\cos(\omega t)d(\omega t) + \frac{2}{\pi}\int_{\omega_s t}^{\pi} u(t)\cos(\omega t)d(\omega t) \\
&= \frac{2}{\pi}\int_0^{\omega_s t} -N\cos(\omega t)d(\omega t) + \frac{2}{\pi}\int_{\omega_s t}^{\pi} N\cos(\omega t)d(\omega t) \\
&= \frac{2}{\pi}\left[-N\sin\theta|_0^{\omega_s t} + N\sin\theta|_{\omega_s t}^{\pi}\right] \\
&= \frac{2N}{\pi}\left[-\sin\omega_s t + 0 + 0 - \sin\omega_s t\right] = -\frac{4N}{\pi}\frac{h}{a}
\end{aligned}$$

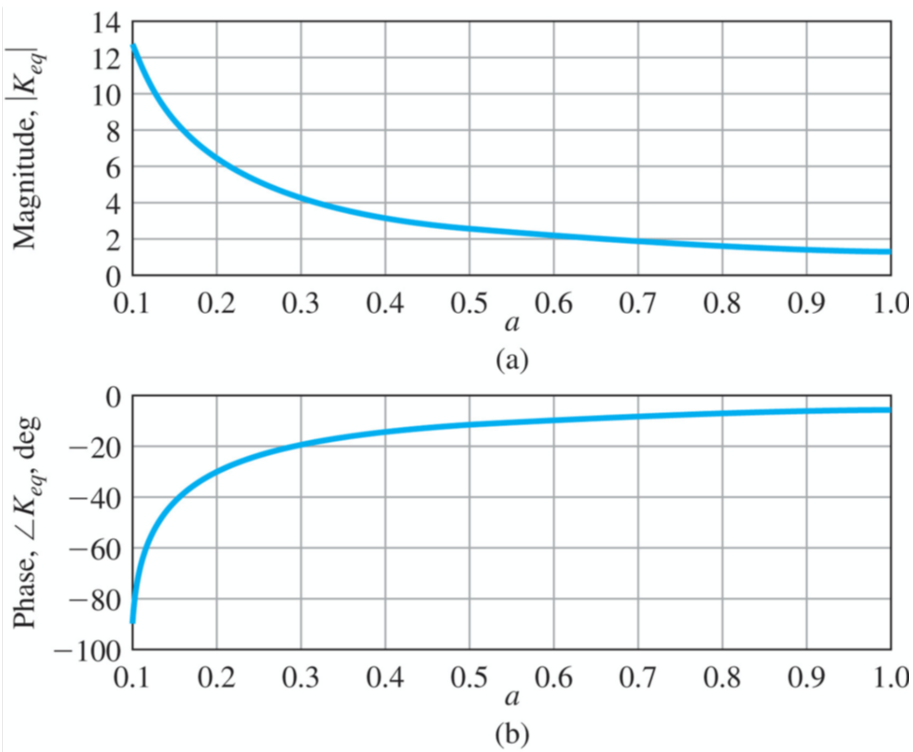
4. Let us calculate b_1 as follow:

$$\begin{aligned}
b_1 &= \frac{2}{\pi}\int_0^{\pi} u(t)\sin(\omega t)d(\omega t) \\
&= \frac{2}{\pi}\int_0^{\omega_s t} u(t)\sin(\omega t)d(\omega t) + \frac{2}{\pi}\int_{\omega_s t}^{\pi} u(t)\sin(\omega t)d(\omega t) \\
&= \frac{2}{\pi}\int_0^{\omega_s t} -N\sin(\omega t)d(\omega t) + \frac{2}{\pi}\int_{\omega_s t}^{\pi} N\sin(\omega t)d(\omega t) \\
&= \frac{2}{\pi}\left[N\cos\theta|_0^{\omega_s t} - N\cos\theta|_{\omega_s t}^{\pi}\right] \\
&= \frac{2N}{\pi}\left[\cos\omega_s t - 1 + 1 + \cos\omega_s t\right] = \frac{4N}{\pi}\sqrt{1 - \frac{h^2}{a^2}}
\end{aligned}$$

5. We finally obtain

$$\begin{aligned} K_{eq}(a) &= \frac{4N}{a\pi}\left[\sqrt{1-\frac{h^2}{a^2}} - j\frac{h}{a}\right] \\ &= \frac{4N}{a\pi} e^{-j\sin^{-1}\frac{h}{a}} \\ &= \frac{4N}{a\pi}\angle - \sin^{-1}\frac{h}{a} \end{aligned}$$

6. The describing function is plotted in Fig. 9.31. The magnitude is proportional to the reciprocal of the input signal amplitude and the phase varies between -90° and 0°.

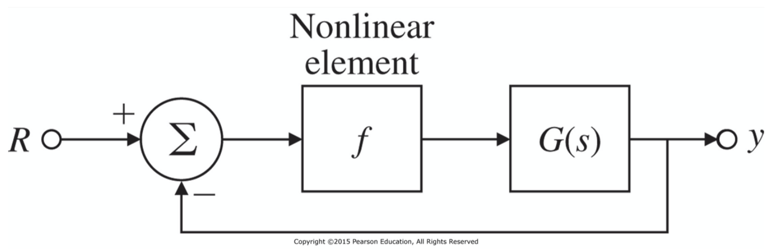


- (9.4.1) Stability Analysis using Describing Functions

  1. Nyquist theorem can be *extended* to deal with nonlinear systems whose nonlinearities have been approximated by describing functions.
  2. With nonlinearity represented by the describing function $K_{eq}(a)$, the characteristic equation is of the form

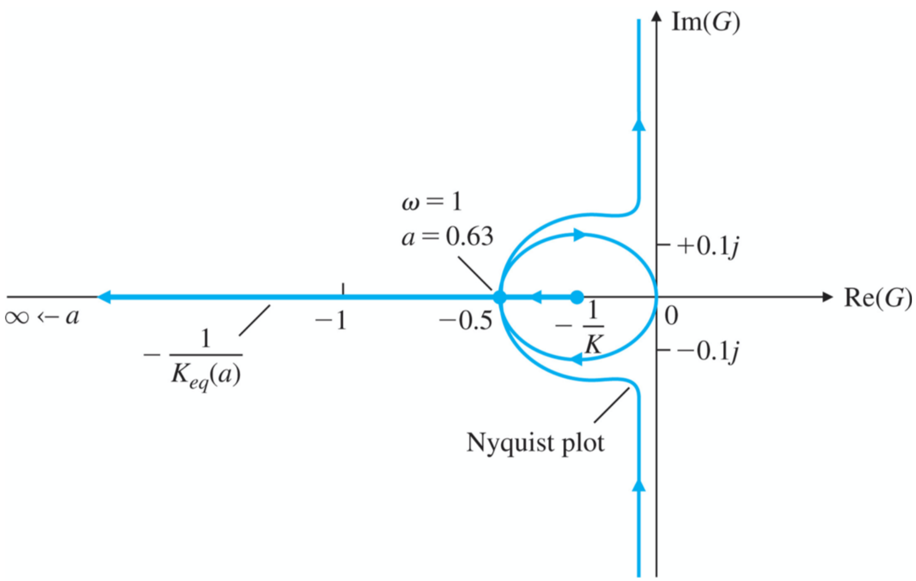
     $$1 + K_{eq}(a)G(s) = 0 \qquad \rightarrow \qquad G(s) = -\frac{1}{K_{eq(a)}}$$

     where $G(s)$ is a loop gain shown in Fig. 9.32.



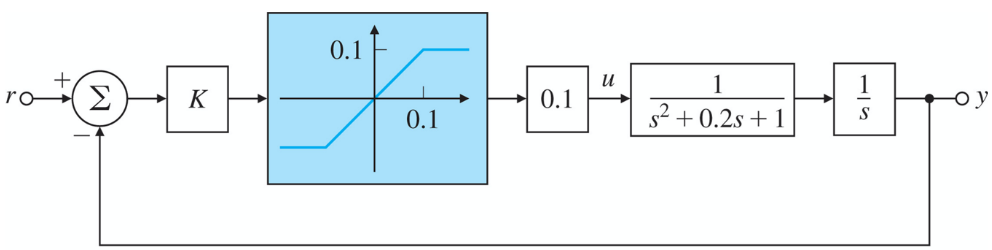
  3. The describing function can be beneficial in determining the conditions under which instability results and can even suggest remedies in eliminating instability.

4. Nyquist plot of the linear loop gain as well as the negative reciprocal of the describing function are superimposed as shown in Fig. 9.33.



5. The point at which they cross corresponds to the *limit cycle* with limit-cycle frequency ω_l and the corresponding amplitude a_l.

6. (Example 9.13) Consider the feedback system in Fig. 9.14. Determine the amplitude and the frequency of the limit cycle using the Nyquist plot?



a) Nyuist plot of $G(s) = \frac{0.1}{s(s^2+0.2s+1)}$ is superimposed with $-\frac{1}{K_{eq}(a)}$ as follow:

$$-\frac{1}{K_{eq}(a)} = -\frac{1}{\frac{2}{\pi}\left(k\sin^{-1}\frac{N}{ak} + \frac{N}{a}\sqrt{1-\frac{N^2}{a^2k^2}}\right)} = -\frac{1}{\frac{2}{\pi}\left(\sin^{-1}\frac{0.1}{a} + \frac{0.1}{a}\sqrt{1-\frac{0.1^2}{a^2}}\right)}$$

where $N = 0.1$ and $k = 1$, which is a straight line that is coincident with negative real axis and parameterized as a functin of input signal amplitude a, as shown in Fig. 9.33

b) The magnitude of $K_{eq} = 0.2$ corresponds to an input amplitude of $a_l = 0..63$. In other words, with $k = 1$ and $N = 0.1$, we have

$$K_{eq}(a_l) = \frac{2}{\pi}\left(\sin^{-1}\frac{0.1}{a_l} + \frac{0.1}{a_l}\sqrt{1-\frac{0.01}{a_l^2}}\right) = 0.2$$

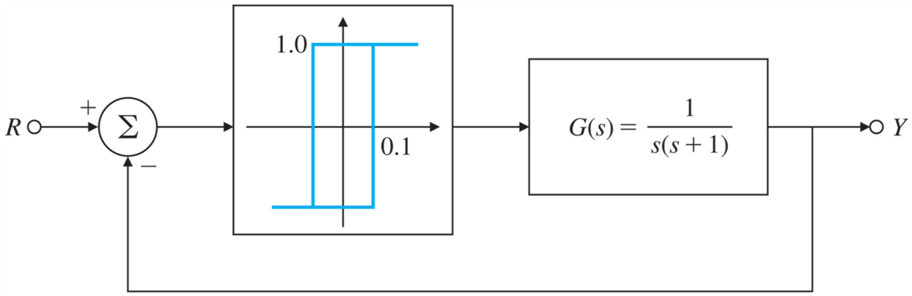
Since

$$\sin^{-1}\frac{0.1}{a_l} \approx \frac{0.1}{a_l}$$

we can approximate the above equation as follows:

$$\frac{1}{a_l} + \frac{1}{a_l}\sqrt{1-\frac{0.01}{a_l^2}} = \pi \quad \rightarrow \quad \pi^2 a_l^4 - 2\pi a_l^3 + 0.01 = 0 \quad \rightarrow \quad a_l = 0.63$$

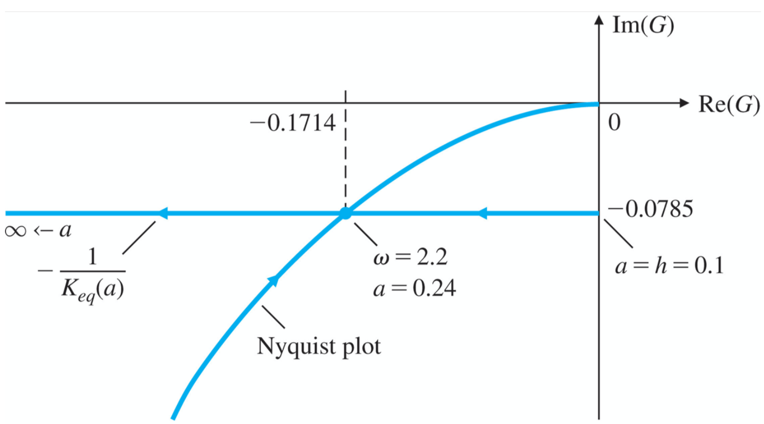
c) The amplitude $a_l = 0.63$ and the frequency $\omega_l = 1$ are in good agreement with our prediction.



7. (Example 9.14) Consider the system with a *hysteresis* nonlinearity. Determine whether the system is stable and find the amplitude and the frequency of the limit cycle?
   a) The Nyquist plot for the system is shown in Fig. 9.36.
   b) The negative reciprocal of the describing function for the hysteresis nonlinearity is

$$-\frac{1}{K_{eq}(a)} = -\frac{1}{\frac{4N}{a\pi}\left[\sqrt{1-\frac{h^2}{a^2}} - j\frac{h}{a}\right]} = -\frac{\pi}{4N}[\sqrt{a^2-h^2}+jh] = -\frac{\pi}{4}[\sqrt{a^2-0.1^2}+j0.1]$$

where $N = 1$ and $h = 0.1$ This is a straight line parallel to the real axis that is parameterized as a function of the input signal amplitude a and is also plotted in Fig. 9.36



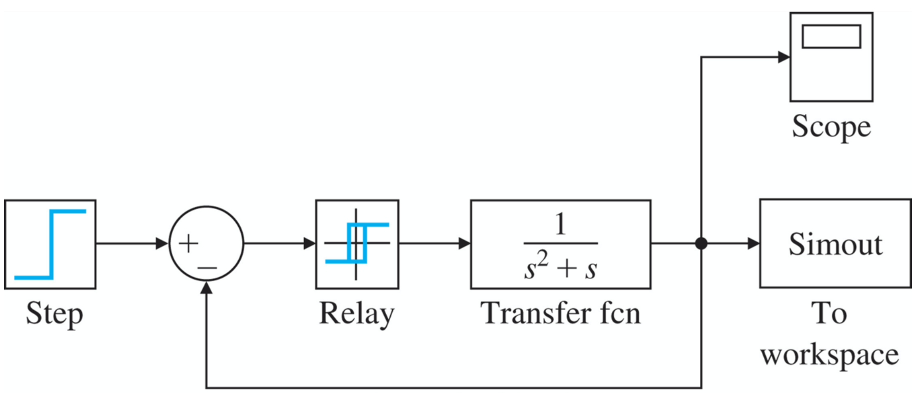
c) We can also determine the limit-cycle information analytically:

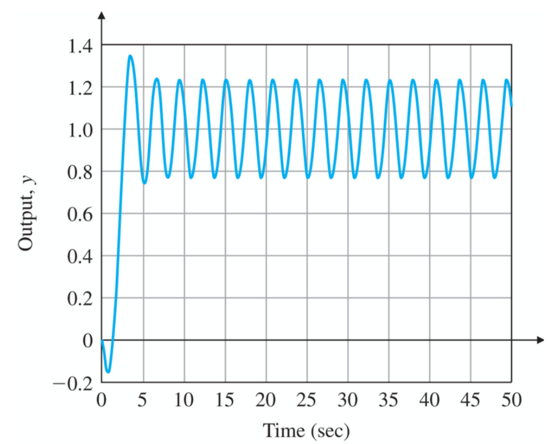
$$-\frac{1}{K_{eq}(a)} = -\frac{\pi}{4}[\sqrt{a^2 - 0.1^2} + j0.1] = G(j\omega) = \frac{1}{j\omega(j\omega+1)} = \frac{1}{-\omega^2 + j\omega}$$

d) By solving above equations, we can get the solutions

$$\omega_l = 2.2 \quad \text{and} \quad a_l = 0.24$$

e) Figs. 9.37 and 9.38 show the simulation diagram and its result. The limit cycle is well predicted by our analysis.





(9장 숙제) 36개의 문제 중 5개 풀어 제출

코로나라는 미증유의 사태에서 1년 동안 제어공학 학습에 열정적으로 임해 주셔서 감사합니다.