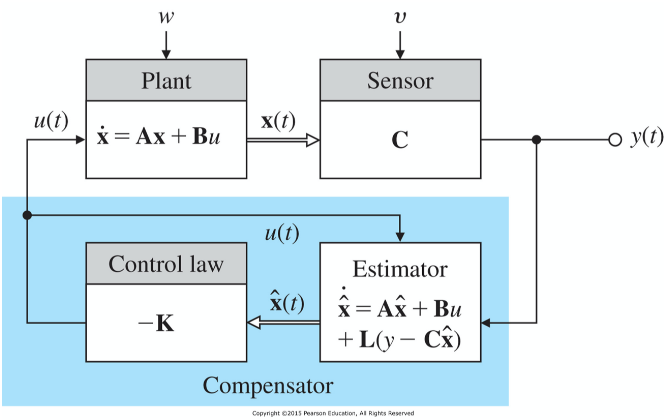
8 Compensator Design: Combined Control Law and Estimator

- If we combine the control law and the estimator, then we have the plant equation with feedback based on the estimated state $u = -K\hat{x}$:

$$\dot{x} = Ax + Bu = Ax - BK\hat{x}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) = A\hat{x} - BK\hat{x} + L(y - C\hat{x})$$

If we introduce the estimation error $\tilde{x} = x - \hat{x}$, then we have



$$\dot{x} = Ax - BK(x - \tilde{x}) = (A - BK)x + BK\tilde{x}$$

$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$

The overall system is described by using the state x and estimation error $\tilde{x}$ as follow:

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

- (Separation Principle) The characteristic equation of the closed-loop system is

$$\det \begin{bmatrix} sI - A + BK & -BK \\ 0 & sI - A + LC \end{bmatrix} = \det(sI - A + BK) \cdot \det(sI - A + LC) = \alpha_c(s) \cdot \alpha_e(s) = 0$$

The designs of the control law and the estimator can be carried out independently.

- (Dynamic Compensator) Consider only controller based on the estimator from y to u

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} - BK\hat{x} + L(y - C\hat{x}) = (A - BK - LC)\hat{x} + Ly \\ u &= -K\hat{x} \end{aligned}$$

The characteristic equation of the dynamic compensator from y to u becomes

$$\det(sI - A + BK + LC) = 0$$

The TF of dynamic compensator will be obtained as

$$D_c(s) = \frac{U(s)}{Y(s)} = -K(sI - A + BK + LC)^{-1}L$$

- (Dynamic Compensator based on Reduced-Order Estimator) The same development can be carried out for the reduced-order estimator. Here the control law is

$$u = -\begin{bmatrix} K_a & K_b \end{bmatrix} \begin{bmatrix} y \\ \hat{x}_b \end{bmatrix} = -K_a y - K_b \hat{x}_b$$

and the reduced-order estimator is

$$\dot{x}_c = (A_{bb} - LA_{ab})x_c + (A_{bb}L - LA_{ab}L + A_{ba} - LA_{aa})y + (B_b - LB_a)u \qquad \hat{x}_b = x_c + Ly$$

Let us apply the control law to the estimator, then we have

$$\begin{aligned} \dot{x}_c &= (A_{bb} - LA_{ab})x_c + (A_{bb}L - LA_{ab}L + A_{ba} - LA_{aa})y + (B_b - LB_a)(-K_a y - K_b \hat{x}_b) \\ &= [A_{bb} - LA_{ab} - (B_b - LB_a)K_b]x_c + [A_{bb}L - LA_{ab}L + A_{ba} - LA_{aa} - (B_b - LB_a)(K_a + K_b L)]y \\ &= A_r x_c + B_r y \\ u &= -K_a y - K_b(x_c + Ly) = [-K_b]x_c + [-(K_a + K_b L)]y \\ &= C_r x_c + D_r y \end{aligned}$$

The TF of the dynamic compensator is

$$D_{cr} = \frac{U(s)}{Y(s)} = C_r(sI - A_r)^{-1}B_r + D_r$$

- (Example) Design a compensator using pole placement for the satellite plant with $G(s) = \frac{1}{s^2}$. Place the control poles at $\omega_n = 1, \zeta = \frac{1}{\sqrt{2}}$ and place the estimator poles at $\omega_n = 5, \zeta = \frac{1}{2}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad\qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Since the desired control poles $s_{1,2} = \omega_n(-\cos\theta \pm j\sin\theta)$ with $\theta = 90^\circ - \sin^{-1}\zeta$, we have

$$s_{1,2} = -\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}} \qquad \rightarrow \qquad \alpha_c(s) = s^2 + \sqrt{2}s + 1 = 0 \qquad \leftarrow \qquad s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Let us find the state feedback gain (matlab code K = place(A,B,pc)) as follows:

$$\det(sI - A + BK) = \det\begin{bmatrix} s & -1 \\ K_1 & s + K_2 \end{bmatrix} = s^2 + K_2 s + K_1 \qquad \rightarrow \qquad K = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}$$

Since the desired estimator poles $s_{1,2} = \omega_n(-\cos\theta \pm j\sin\theta)$ with $\theta = 90^\circ - \sin^{-1}\zeta$, we have

$$s_{1,2} = 5(-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}) \qquad \rightarrow \qquad \alpha_e(s) = s^2 + 5s + 25 = 0 \qquad \leftarrow \qquad s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

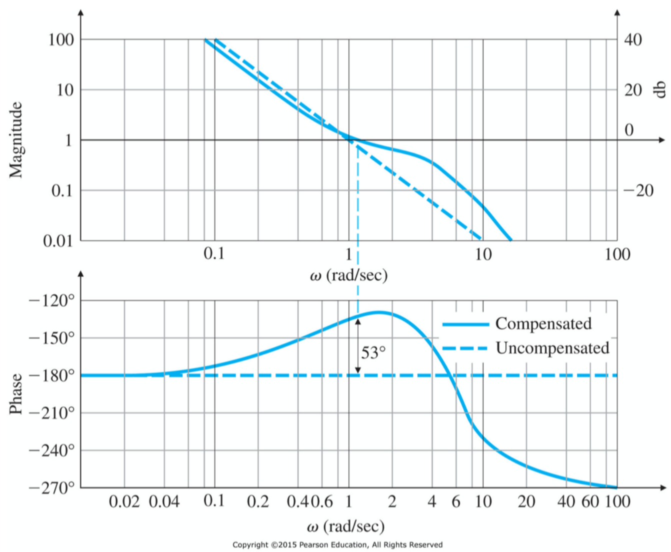
Let us find the estimator gain (matlab code Lt = place(A',B',pe), L = Lt') as follows:

$$\det(sI - A + LC) = \det\begin{bmatrix} s + L_1 & -1 \\ L_2 & s \end{bmatrix} = s^2 + L_1 s + L_2 \qquad \rightarrow \qquad L = \begin{bmatrix} 5 \\ 25 \end{bmatrix}$$

The TF of dynamic compensator is

$$
\begin{aligned}
D_c(s) &= -K(sI - A + BK + LC)^{-1}L \\
&= -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} s+L_1 & -1 \\ K_1+L_2 & s+K_2 \end{bmatrix}^{-1} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \\
&= -\begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} s+5 & -1 \\ 26 & s+\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 25 \end{bmatrix} \\
&= -40.4\frac{s+0.619}{s^2+6.414s+33.071}
\end{aligned}
$$

which looks very much like a lead compensator in that it has a zero -0.619 on the real axis to the right of its poles $-3.21 \pm j4.77$. Phase margin can be checked from Nyquist plot of $G(s)D_c(s) = 40.4\frac{s+0.619}{s^2(s^2+6.414s+33.071)}$,



- (Example) Repeat the design for $G(s) = \frac{1}{s^2}$, but use a reduced-order estimator. Place the one estimator pole at -5

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \qquad K = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}$$

The characteristic equation of the reduced-order estimator and the desired characteristic equation are

$$\det[s - A_{bb} + LA_{ab}] = s + L \qquad \rightleftarrows \qquad s + 5 = 0$$

Thus $L = 5$. From the above results, we have

$$\begin{aligned} A_r &= A_{bb} - LA_{ab} - (B_b - LB_a)K_b = 0 - 5 - \sqrt{2} = -6.414 \\ B_r &= A_r L + A_{ba} - LA_{aa} - (B_b - LB_a)K_a = -6.414 \times 5 - 1 = -33.07 \\ C_r &= -K_b = -1.414 \\ D_r &= -K_a - K_b L = -1 - \sqrt{2} \times 5 = -8.071 \end{aligned}$$

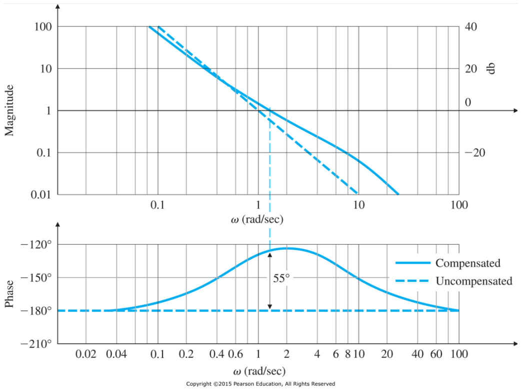
Thus the dynamic compensator becomes

$$\begin{aligned} \dot{x}_c &= -6.414 x_c - 33.07 y \\ u &= -1.414 x_c - 8.071 y \end{aligned}$$

The TF of the dynamic compensator is

$$D_{cr}(s) = \frac{U(s)}{Y(s)} = C_r(sI - A_r)^{-1}B_r + D_r = -8.071\frac{s + 0.619}{s + 6.414}$$

which is precisely a lead compensator. Phase margin of $55°$ can be confirmed from Nyquist plot.



- (Example 7.27) Design a compensator using pole placement for the satellite plant with $G(s) = \frac{1}{s^2}$. Place the control poles at $\omega_n = 1.13, \zeta = 0.7$ and place the estimator poles at $\omega_n = 8, \zeta = 0.5$
- (Example 7.28) Repeat the design for $G(s) = \frac{1}{s^2}$, but use a reduced-order estimator. Place the one estimator pole at -10
- (Example 7.29) Dynamic compensator using full-order estimator can be unstable. However, an unstable compensator is typically not acceptable.
- (Example 7.30) Dynamic compensator using reduced-order estimator can be nonminimum phase.
- (Example 7.31) Dynamic compensator using the SRL and LQR can be stable and minimum phase, but it is not guaranteed.
- (Example 7.32) The control gains are much lower, and the compensator design is less radical.