7 Estimator Design

- The control law designed in the previous section assumed that all the state-variables are available for feedback.
- However, in most cases, not all the state-variables are measured.
- How to reconstruct all the state-variables of the system from a few measurement is suggested
- If the estimate of the state denoted by $\hat{x}$ is used for the control law,

$$u = -K\hat{x} + \bar{N}r \qquad \leftarrow \qquad u = -Kx + \bar{N}r$$

- (7.7.1) Full-Order Estimators

1. One method of estimating the state is to construct a full-order model of the plant dynamics

$$\dot{\hat{x}} = A\hat{x} + Bu(t) \qquad \leftarrow \qquad \dot{x} = Ax + Bu$$

   where A, B and $u(t)$ are known. Let us define the error in the estimator $\tilde{x} = x - \hat{x}$, then the dynamics of this error system is given by

$$\dot{\tilde{x}} = A\tilde{x} \qquad\qquad \tilde{x}(0) = x(0) - \hat{x}(0)$$

   if A is stable, the estimation error $\tilde{x} \to 0$ as $t \to \infty$

2. In order to influence the rate at which the state estimate converges to the true state, use the feedback $L(y - \hat{y}) = LC(x - \hat{x})$ in the estimator:

$$\dot{\hat{x}} = A\hat{x} + Bu(t) + L(y - \hat{y}) = A\hat{x} + Bu(t) + LC(x - \hat{x})$$

   where $L = [l_1, l_2, \cdots, l_n]^T$. Let us find the estimation error dynamics and characteristic equation:

$$\dot{\tilde{x}} = (A - LC)\tilde{x} \qquad\qquad \det[sI - A + LC] = 0$$

3. If we can choose L so that $A - LC$ has stable and reasonably fast eigenvalues, $\tilde{x}$ will decay to zero. If the desired estimator characteristic equation is

$$\alpha_e(s) = (s - \beta_1)(s - \beta_2)\cdots(s - \beta_n) = \det[sI - A + LC]$$

   then L can be calculated by comparing the both characteristic equations.

4. (Example 7.24) Design an estimator of the system. Compute the estimator gain matrix that will place both the estimator error poles at $-10\omega_o$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad\qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

The desired estimator characteristic equation is

$$\alpha_e(s) = (s + 10\omega_o)^2 = s^2 + 20\omega_o s + 100\omega_o^2$$

The estimator and the error dynamics become

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \qquad \text{and} \qquad \dot{\tilde{x}} = (A - LC)\tilde{x}$$

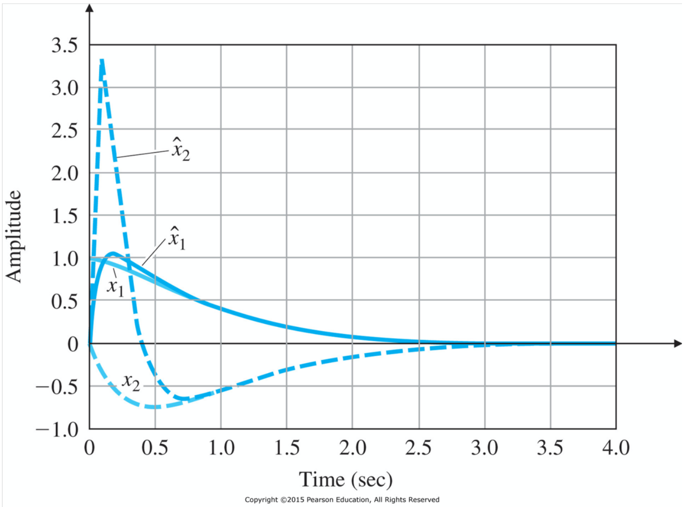
Therefore the characteristic equation of the error dynamics becomes

$$\det[sI - A + LC] = \det\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\right\} = \det\left\{\begin{bmatrix} s + l_1 & -1 \\ \omega_o^2 + l_2 & s \end{bmatrix}\right\} = s^2 + l_1 s + l_2 + \omega_o^2$$

Thus we have

$$L = \begin{bmatrix} 20\omega_o \\ 99\omega_o^2 \end{bmatrix}$$

Note that the estimation error decays approximately five times (desired estimator pole $-10\omega_o$) faster than the decay of the controlled state itself (desired control pole $-2\omega_o$), as we designed it to do.



5. (OCF, Observer Canonical Form) is dual form of the control canonical form (CCF). Thus we have

$$\dot{x}_o = A_o x_o + B_o u \qquad\qquad y = C_o x_o$$

where

$$A_o = A_c^T = \begin{bmatrix} -a_1 & 1 & 0 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & 1 \\ -a_n & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad B_o = C_c^T = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$C_o = B_c^T = \begin{bmatrix} 1 & 0 & 0 & \cdots 0 \end{bmatrix} \qquad D_o = D_c = 0$$

Like the CCF, the OCF can be obtained by reading the coefficients of the TF.

6. One of the advantages of OCF is that the estimator gains can be obtained from it by inspection. For example

$$A_o - LC_o = \begin{bmatrix} -a_1 - l_1 & 1 & 0 \\ -a_2 - l_2 & 0 & 1 \\ -a_3 - l_3 & 0 & 0 \end{bmatrix}$$

which has the characteristic equation:

$$s^3 + (a_1 + l_1)s^2 + (a_2 + l_2)s + (a_3 + l_3) = 0$$

In other words, the estimator gain can be found by comparing the coefficients of characteristic equation with $\alpha_e(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$ such as

$$l_1 = \alpha_1 - a_1$$

$$l_2 = \alpha_2 - a_2$$

$$l_3 = \alpha_3 - a_3$$

7. Observability refers to our ability to deduce information about all the modes of the system by monitoring only the sensed outputs.
8. Unobservability results when some mode or subsystem is disconnected physically from the output and therefore no longer appears in the measurements.
9. The mathematical test for determining observability is that the observability matrix must have independent columns (full rank).

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \leftarrow \qquad \mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

10. As with control-law design, the Ackermann's formula for the estimator can be used as

$$L = \alpha_e(A)\mathcal{O}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad \leftarrow \qquad K = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}\alpha_c(A)$$

- (7.7.2) Reduced-Order Estimators

  1. The reduced-order estimator reduces the order of the estimator by the number of the sensed (measurement) output.
  2. Let us assume that the output equals the first state as $y = x_a$, which is directly measured, and x_b, which represents the remaining state-variables that need to be estimated.
  3. Let us partition the state-space representation (for example, OCF) as follows:

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_b \end{bmatrix} = \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix}$$

The dynamics of the output $y = x_a$ and the unmeasured state-variables x_b are given by

$$\dot{x}_b = A_{ba}y + A_{bb}x_b + B_b u \qquad \dot{y} = A_{aa}y + A_{ab}x_b + B_a u$$

If we rearrange the above equations in such a way that the known terms are collected into the parenthesis, then we have

$$\dot{x}_b = A_{bb}x_b + (A_{ba}y + B_b u) \qquad (\dot{y} - A_{aa}y - B_a u) = A_{ab}x_b$$

$$\dot{x}_b = A_{bb}x_b + u_t \qquad y_t = A_{ab}x_b$$

where the known input $u_t = A_{ba}y + B_b u$ and the known measurement $y_t = \dot{y} - A_{aa}y - B_a u$.

4. Design the reduced-order estimator as follow:

$$\begin{aligned}\dot{\hat{x}}_b &= A_{bb}\hat{x}_b + u_t + L(y_t - A_{ab}\hat{x}_b)\\ &= A_{bb}\hat{x}_b + (A_{ba}y + B_b u) + L((\dot{y} - A_{aa}y - B_a u) - A_{ab}\hat{x}_b)\\ &= (A_{bb} - LA_{ab})\hat{x}_b + (A_{ba} - LA_{aa})y + (B_b - LB_a)u + L\dot{y}\end{aligned}$$

If $\dot{y}$ can be removed by letting the new control state $x_c = \hat{x}_b - Ly$, then we have

$$\begin{aligned}\dot{x}_c &= (A_{bb} - LA_{ab})(x_c + Ly) + (A_{ba} - LA_{aa})y + (B_b - LB_a)u\\ &= (A_{bb} - LA_{ab})x_c + (A_{bb}L - LA_{ab}L + A_{ba} - LA_{aa})y + (B_b - LB_a)u\\ \hat{x}_b &= x_c + Ly\end{aligned}$$

5. The characteristic equation of the reduced-order estimator becomes

$$\det[sI - A_{bb} + LA_{ab}] = 0$$

6. (Example 7.25) Design a reduced-order estimator with the estimation error pole at $-10\omega_o$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\omega_o^2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u \qquad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Since $y = x_1$, the reduced-order estimator becomes

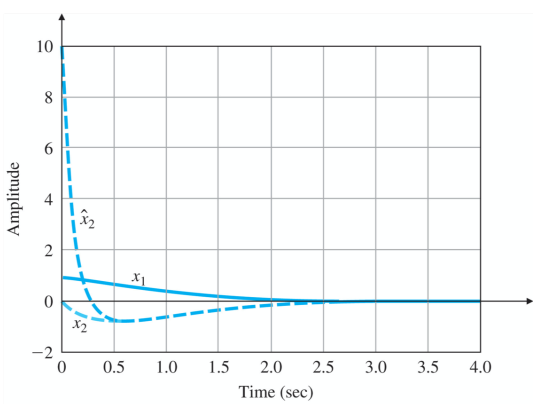
$$\dot{x}_c = (0 - L)x_c + (0 - L^2 - \omega_o^2)y + (1 - 0)u = -Lx_c - (L^2 + \omega_o^2)y + u \qquad \hat{x}_b = x_c + Ly$$

The characteristic equations of the above and the desired are

$$s + L = 0 \qquad s + 10\omega_o = 0$$

Thus $L = 10\omega_o$. The reduced-order estimator is obtained as

$$\dot{x}_c = -10\omega_o x_c - 101\omega_o^2 y + u \qquad \hat{x}_b = x_c + 10\omega_o y$$



- (7.7.3) Estimator Pole Selection

1. The estimator poles can be chosen to be faster than controller poles by a factor of 2 to 6.
2. In optimal estimation theory, the best choice for estimator gain is dependent on the ratio of sensor noise intensity v to process (disturbance) noise intensity w
3. This is best understood by re-examining the estimator equation:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

   to see how it interacts with the system when process noise w is present.
4. The plant with process noise and the measurement equation with sensor noise are described by

$$\dot{x} = Ax + Bu + B_1 w \qquad\qquad y = Cx + v$$

   The estimation error ($\tilde{x} = x - \hat{x}$) equation becomes

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + B_1 w - Lv$$

   where the sensor noise is multiplied by L and the process noise is not. Big estimator gain L will bring a noise amplification.
5. The estimator SRL equation is

$$1 + qG_e(-s)G_e(s) = 0 \qquad \leftarrow \qquad G_e(s) = C(sI - A)^{-1}B_1$$

   where q is the ratio of input disturbance noise intensity to sensor noise intensity, and G_e is the TF from the process noise to the sensor output equation.

6. (Example 7.26) Draw the estimator SRL for the following system. Take the output to be a noisy measurement of position with noise intensity ratio q:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v$$

The TF from w to y becomes

$$G_e(s) = \frac{1}{s^2 + \omega_o^2}$$

The SRL is obtained as $q : 0 \to \infty$

$$1 + qG_e(-s)G_e(s) = 1 + q\frac{1}{(s^2 + \omega_o^2)(s^2 + \omega_o^2)} = 0 \qquad \to \qquad s^4 + 2\omega_o^2 s^2 + \omega_o^4 + q = 0$$

We could choose two stable roots for a given value of q, for example, $s = -3 \pm j3.18$ for $q = 365$.