4. Properties of Laplace Transforms (LT) (see Table A.1 in Appendix A (page 866))

a) Superposition

$$\begin{aligned}
\mathcal{L}[\alpha f_1(t) + \beta f_2(t)] &= \int_0^\infty (\alpha f_1(t) + \beta f_2(t)) e^{-st} dt \\
&= \alpha \int_0^\infty f_1(t) e^{-st} dt + \beta \int_0^\infty f_2(t) e^{-st} dt \\
&= \alpha F_1(s) + \beta F_2(s)
\end{aligned}$$

b) Time Delay $f_1(t) = t(t - \lambda)$ with a time delay of λ

$$\begin{aligned}
F_1(s) &= \int_0^\infty f(t-\lambda) e^{-st} dt \qquad \textbf{with} \quad \eta = t - \lambda \\
&= \int_0^\infty f(\eta) e^{-s(\lambda+\eta)} d\eta \\
&= e^{-\lambda s} \int_0^\infty f(\eta) e^{-s\eta} d\eta \\
&= e^{-\lambda s} F(s)
\end{aligned}$$

c) Time Scaling $f_1(t) = f(at)$ with a scaling factor a

$$\begin{aligned}
F_1(s) &= \int_0^\infty f(at) e^{-st} dt \qquad \textbf{with} \quad \eta = at \\
&= \int_0^\infty f(\eta) e^{-\frac{s\eta}{a}} \frac{1}{a} d\eta \qquad \textbf{with} \quad s' = \frac{s}{a} \\
&= \frac{1}{a} F(s') = \frac{1}{a} F\left(\frac{s}{a}\right)
\end{aligned}$$

d) Shift in Frequency $f_1(t) = e^{-at} f(t)$

$$\begin{aligned} F_1(s) &= \int_0^\infty e^{-at} f(t) e^{-st} dt \\ &= \int_0^\infty f(t) e^{-(s+a)t} dt \qquad \textbf{with} \quad s' = s + a \\ &= F(s') \\ &= F(s+a) \end{aligned}$$

e) Differentiation

$$\begin{aligned} \mathcal{L}[\ddot{f}(t)] &= \int_0^\infty \ddot{f}(t) e^{-st} dt = \int_0^\infty e^{-st} \ddot{f}(t) dt \\ &= e^{-st} \dot{f}(t)\Big|_0^\infty - (-s) \int_0^\infty e^{-st} \dot{f}(t) dt \\ &= e^{-st} \dot{f}(t)\Big|_0^\infty + s \left[e^{-st} f(t)\big|_0^\infty - (-s) \int_0^\infty e^{-st} f(t) dt \right] \\ &= 0 - \dot{f}(0) + s\left[0 - f(0) + sF(s)\right] \\ &= s^2 F(s) - s f(0) - \dot{f}(0) \\ \mathcal{L}[f^{(m)}(t)] &= s^m F(s) - s^{m-1} f(0) - s^{m-2} \dot{f}(0) - \cdots - f^{(m-1)}(0) \end{aligned}$$

where $f^{(m)}(t)$ denotes the mth derivative w.r.t. time

f) Integration $f_1(t) = \int_0^t f(\eta)d\eta$

$$\begin{aligned} F_1(s) &= \int_0^\infty \left[\int_0^t f(\eta)d\eta\right] e^{-st}dt \\ &= \left[\int_0^t f(\eta)d\eta\right] \frac{e^{-st}}{-s}\Bigg|_0^\infty - \int_0^\infty f(t)\frac{e^{-st}}{-s}dt \\ &= \frac{1}{s}F(s) \end{aligned}$$

g) Convolution $f_1(t) \star f_2(t) = \int_0^t f_1(t-\tau)f_2(\tau)d\tau$

$$\mathcal{L}[f_1(t) \star f_2(t)] = F_1(s)F_2(s)$$

h) Time Product

$$\mathcal{L}[f_1(t)f_2(t)] = \frac{1}{2\pi j}[F_1(s) \star F_2(s)]$$

i) Multiplication by Time $f_1(t) = tf(t)\ :\ F_1(s) = \mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$

$$\begin{aligned} \frac{d}{ds}F(s) &= \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt \\ &= \int_0^\infty f(t)(-t)e^{-st}dt \\ &= -\int_0^\infty [tf(t)]e^{-st}dt \qquad \textbf{with} \quad f'(t) = tf(t) \\ &= -\mathcal{L}[f'(t)] = -\mathcal{L}[tf(t)] \end{aligned}$$

5. Inverse Laplace Transform (LT) by Partial-Fraction Expansion

- Consider TF

$$\begin{aligned} F(s) &= \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_m s + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \\ &= K \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \\ &= \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \cdots + \frac{C_n}{s-p_n} \end{aligned}$$

  where $s = z_i$ and $s = p_i$ are referred to as a zero and a pole of the TF, respectively.

- By multiplying both sides by the factor $(s - p_1)$, we can get C_1 term as follow:

$$(s-p_1)F(s) = C_1 + C_2 \frac{s-p_1}{s-p_2} + \cdots + C_n \frac{s-p_1}{s-p_n} \qquad \rightarrow \qquad C_1 = (s-p_1)F(s)|_{s=p_1}$$

  Thus ith coefficient can be expressed in a similar form:

$$C_i = (s-p_i)F(s)|_{s=p_i} \qquad \text{for} \quad i = 1, 2, 3, \cdots, n$$

  where it is called the cover-up method.

(Example 3.11, Partial-Fraction Expansion) Find $y(t)$ from

$$\begin{aligned} Y(s) &= \frac{(s+2)(s+4)}{s(s+1)(s+3)} \\ &= \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+3} \end{aligned}$$

where

$$\begin{aligned} C_1 &= \left.\frac{(s+2)(s+4)}{(s+1)(s+3)}\right|_{s=0} = \frac{8}{3} \\ C_2 &= \left.\frac{(s+2)(s+4)}{s(s+3)}\right|_{s=-1} = -\frac{3}{2} \\ C_3 &= \left.\frac{(s+2)(s+4)}{s(s+1)}\right|_{s=-3} = -\frac{1}{6} \end{aligned}$$

The solution is obtained as follows:

$$\therefore \qquad y(t) = \frac{8}{3} - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t} \qquad \textbf{for} \quad t \geq 0$$

6. The Final Value Theorem

- Consider the LT of differentiation

$$\begin{aligned}
\int_0^\infty \dot{y}(t)e^{-st}dt &= sY(s) - y(0) \\
\lim_{s\to 0}\int_0^\infty \dot{y}(t)e^{-st}dt &= \lim_{s\to 0}[sY(s) - y(0)] \\
\int_0^\infty \dot{y}(t)dt &= \lim_{s\to 0}[sY(s) - y(0)] \\
y(\infty) - y(0) &= \lim_{s\to 0}[sY(s) - y(0)] \\
y(\infty) &= \lim_{s\to 0} sY(s)
\end{aligned}$$

- If all poles of $sY(s)$ are in the left half of the s-plane (or if $Y(s)$ is stable), then

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

(Example 3.12) Find the final value $y(\infty)$?

$$
\begin{aligned}
Y(s) &= \frac{3(s+2)}{s(s^2+2s+10)} \\
y(\infty) &= \lim_{s\to 0} \frac{3(s+2)}{s^2+2s+10} \\
&= \frac{6}{10} \\
&= 0.6
\end{aligned}
$$

(Example 3.13) Find the final value $y(\infty)$?

$$
\begin{aligned}
Y(s) &= \frac{3}{s(s-2)} \\
y(\infty) &\neq \lim_{s\to 0} \frac{3}{s-2} = -\frac{3}{2} = -1.5
\end{aligned}
$$

because the final value theorem is applied to the stable system, namely, in the case that all poles are located on the left-hand side.
For example,

$$
\begin{aligned}
Y(s) &= \frac{3}{s(s-2)} = \frac{-1.5}{s} + \frac{1.5}{s-2} \\
y(t) &= -1.5 + 1.5e^{2t} \quad \textbf{for} \quad t \geq 0 \\
y(\infty) &= \infty
\end{aligned}
$$

- **DC gain is defined as the final value of the unit-step response for stable systems ($Y(s) = G(s)U(s) = G(s)\frac{1}{s}$)**

$$\textbf{DC gain} = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\left[G(s)\frac{1}{s}\right] = \lim_{s\to 0} G(s)$$

(Example 3.14, DC Gain) Find the DC gain of the following TF

$$G(s) = \frac{3(s+2)}{s^2+2s+10}$$

$$\textbf{DC gain} = \lim_{s\to 0} G(s) = 0.6$$

7. Using Laplace Transform (LT) to Solve Differential Equation (DE)
(Example 3.15 Homogeneous DE) Find the solution of DE

$$\ddot{y}(t) + y(t) = 0, \qquad \text{where} \;\; y(0) = \alpha \;\; \dot{y}(0) = \beta$$

$$\begin{aligned}
s^2Y(s) - y(0)s - \dot{y}(0) + Y(s) &= 0 \\
(s^2+1)Y(s) &= \alpha s + \beta \\
Y(s) &= \frac{\alpha s + \beta}{s^2+1} \\
Y(s) &= \alpha\frac{s}{s^2+1} + \beta\frac{1}{s^2+1} \\
y(t) &= \alpha\cos t + \beta\sin t \qquad \text{for} \;\; t \geq 0
\end{aligned}$$

(Example 3.16 Forced DE) Find the solution of DE

$$\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = 3 \cdot 1(t), \qquad \textbf{where} \quad y(0) = \alpha \quad \dot{y}(0) = \beta$$

$$\begin{aligned}
[s^2Y(s) - y(0)s - \dot{y}(0)] + 5[sY(s) - y(0)] + 4Y(s) &= \frac{3}{s} \\
(s^2 + 5s + 4)Y(s) &= \frac{3}{s} + \alpha s + (\beta + 5\alpha) \\
Y(s) &= \frac{\alpha s^2 + (\beta + 5\alpha)s + 3}{s(s+1)(s+4)} \\
Y(s) &= \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+4}
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \frac{3}{4} \\
C_2 &= \frac{4\alpha + \beta - 3}{3} \\
C_3 &= \frac{3 - 4\alpha - 4\beta}{12}
\end{aligned}$$

Thus

$$y(t) = C_1 + C_2 e^{-t} + C_3 e^{-4} \qquad \textbf{for} \quad t \geq 0$$

(Example 3.17 Forced Solution with Zero Initial Conditions) Find the solution of DE

$$\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = 2e^{-2t} \cdot 1(t), \qquad \textbf{where} \quad y(0) = 0 \quad \dot{y}(0) = 0$$

$$\begin{aligned} s^2Y(s) + 5sY(s) + 4Y(s) &= \frac{2}{s+2} \\ (s^2 + 5s + 4)Y(s) &= \frac{2}{s+2} \\ Y(s) &= \frac{2}{(s+1)(s+2)(s+4)} \\ Y(s) &= \frac{C_1}{s+1} + \frac{C_2}{s+2} + \frac{C_3}{s+4} \end{aligned}$$

where

$$\begin{aligned} C_1 &= \frac{2}{3} \\ C_2 &= -1 \\ C_3 &= \frac{1}{3} \end{aligned}$$

Thus

$$y(t) = \frac{2}{3}e^{-t} - e^{-2t} + \frac{1}{3}e^{-4t} \qquad \textbf{for} \quad t \geq 0$$

8. Poles and Zeros

- Consider a rational TF as two kinds of form

$$
\begin{aligned}
H(s) &= \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_m s + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \\
&= K \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}
\end{aligned}
$$

- If $s = z_i$, then

$$H(s)|_{s=z_i} = 0$$

The zeros also correspond to the signal transmission blocking properties of the system and are also called the transmission zeros of the system.

- If $s = p_i$, then

$$H(s)|_{s=p_i} = \infty$$

The poles of the system determine its stability properties.

- Consider a rational TF as two kinds of form

$$H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_m s + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \quad \rightarrow \quad \lim_{s\to\infty} H(s) = \frac{b_1}{s^{n-m}}$$

$$= K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

- The system is said to have $n-m$ zeros at infinity if $m<n$ because the TF approaches zero as s approaches infinity. $\rightarrow$ The system is said to be strictly proper
- No physical system can have $n<m$; otherwise it would have an infinite response at $\omega=\infty$. $\rightarrow$ The system is said to be non-proper
- If $z_i = p_j$, then there are cancellations in the TF. $\rightarrow$ It may lead to undesirable properties.

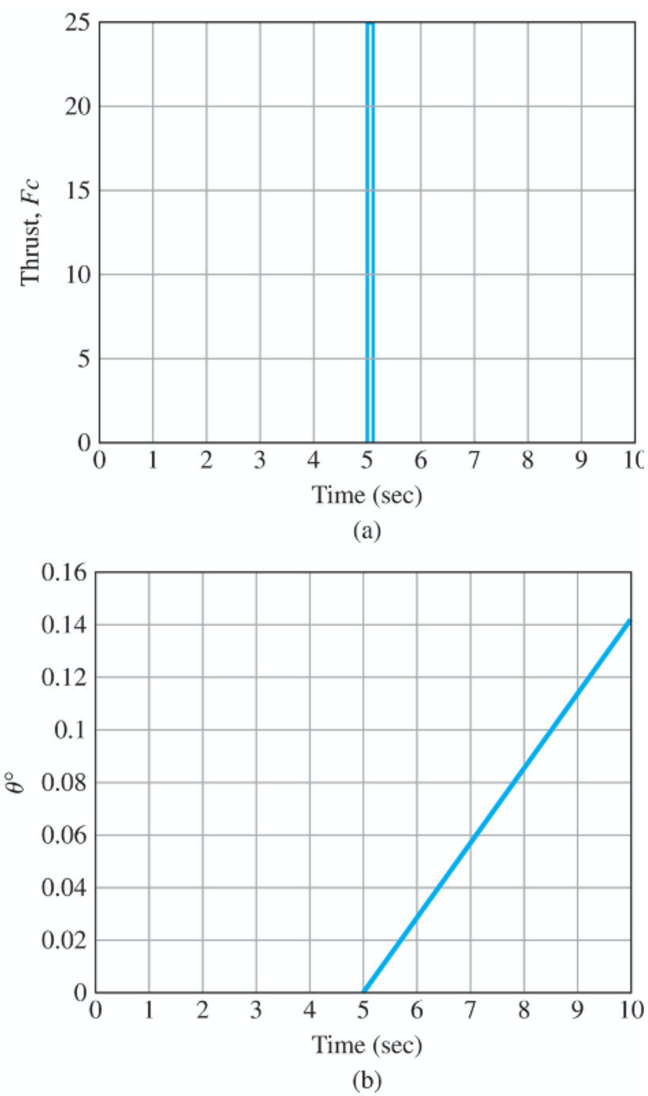
9. Linear System Analysis using MATLAB
(Example 3.18), Matlab of (Example 2.1)

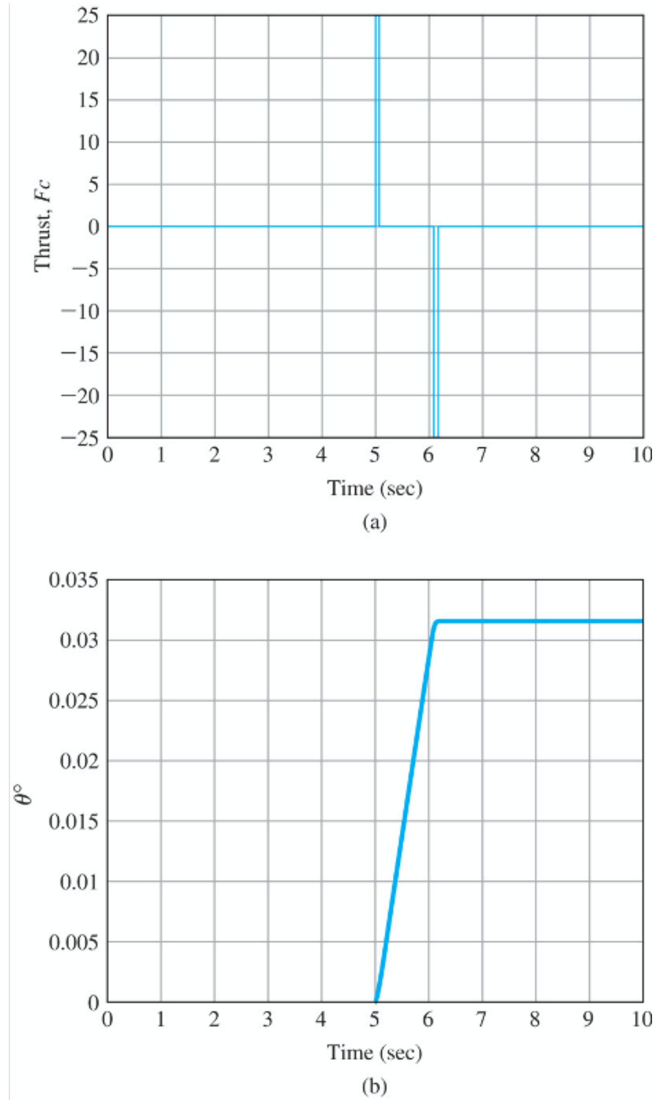
```
num = [0 0 0.001]
den = [1 0.05 0]
[z,p,k] = tf2zp(num,den)
```

(Example 3.21) Matlab of (Example 2.3)

```
s = tf('s')
sysG = 0.0002/s^2
t = 0:0.01:10
u1 = [zeros(1,500) 25*ones(1,10) zeros(1,491)]
[y1] = lsim(sysG, u1,t)
y1 = y1*(180/pi)
plot(t,u1)
plot(t,y1)

u2 = [zeros(1,500) 25*ones(1,10) zeros(1,100) -25*ones(1,10) zeros(1,381)]
[y2] = lsim(sysG, u2,t)
y2 = y2*(180/pi)
plot(t,u2)
plot(t,y2)
```


Thrust, Fc
25
20
15
10
5
0
0 1 2 3 4 5 6 7 8 9 10
Time (sec)
(a)
θ°
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0 1 2 3 4 5 6 7 8 9 10
Time (sec)
(b)


Thrust, Fc
25
20
15
10
5
0
−5
−10
−15
−20
−25
0 1 2 3 4 5 6 7 8 9 10
Time (sec)
(a)
θ°
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0 1 2 3 4 5 6 7 8 9 10
Time (sec)
(b)