3 Selected Illustrative Root Loci

- For the double integrator system of $G(s) = \frac{1}{s^2}$, if we form a unity feedback system with this plant and a proportional controller, draw the root locus with respect to controller gain $K = k_P$?

  The characteristic equations becomes

$$1 + k_P \frac{1}{s^2} = 0$$

1. (Rule 1, Start and End)

$$\text{when } k_P = 0, \qquad s = 0, 0 \quad \text{poles of } L(s) = \tfrac{1}{s^2}$$
$$\text{when } k_P = \infty, \qquad s = \infty, \infty \quad \text{zeros of } L(s)$$

2. (Rule 2, Real Axis) No root on the real axis
3. (Rule 3, Asymptotes)

$$\phi_l = \frac{180^\circ + 360(l-1)}{2} = \pm 90^\circ \qquad\qquad \alpha = \frac{0}{2} = 0$$

4. (Rule 4, Departure Angles and Arrival Angles)

$$2\phi_{l,dep} = -180^\circ - 360(l-1) \quad \rightarrow \quad \phi_{dep} = \pm 90^\circ$$

5. (Rule 5, Break-in and Breakaway Points)

$$\frac{dk_P}{ds} = -2s = 0 \quad \rightarrow \quad s = 0 \ \text{ breakaway point}$$

- (Example 5.3, Root Locus for Satellite Attitude Control with PD Control) For the double integrator system of $G(s) = \frac{1}{s^2}$, if we form a unity feedback system with this plant and a PD controller satisfying $k_P = k_D > 0$, draw the root locus with respect to controller gain $K = k_D$

$$1 + (k_P + k_D s)\frac{1}{s^2} = 0 \qquad \rightarrow \qquad 1 + K\frac{s+1}{s^2} = 0 \qquad \text{with} \quad K = k_P = k_D > 0$$

1. (Rule 1, Start and End)

$$\text{when } K = 0, \qquad s = 0, 0 \qquad \text{poles of } L(s) = \tfrac{s+1}{s^2}$$
$$\text{when } K = \infty, \qquad s = -1, \infty \qquad \text{zeros of } L(s)$$

2. (Rule 2, Real Axis) Negative real axis of $s \leq -1$ is locus
3. (Rule 3, Asymptotes)
4. (Rule 4, Departure Angles and Arrival Angles)

$$2\phi_{l,dep} = \angle(0 - (-1)) - 180^\circ - 360^\circ(l-1) \qquad \rightarrow \qquad \phi_{dep} = \pm 90^\circ$$

5. (Rule 5, Break-in and Breakaway Points)

$$\frac{dK}{ds} = -\frac{d}{ds}\frac{s^2}{s+1} = -\frac{2s(s+1) - s^2}{(s+1)^2} = -\frac{s(s+2)}{(s+1)^2} = 0 \qquad \rightarrow \qquad s = 0 \ \text{ breakaway} \qquad s = -2 \quad \text{break-in}$$

6. For the understanding of locus of the complex roots, let us apply $s = \sigma + j\omega$ for $0 < K < 4$ because $K = 4$ at break-in point $s = -2$:

$$s^2 + Ks + K = \sigma^2 - \omega^2 + 2j\sigma\omega + K(\sigma + j\omega) + K = 0 \quad \rightarrow \quad \sigma^2 - \omega^2 + K\sigma + K = 0 \text{ and } 2\sigma\omega + K\omega = 0$$

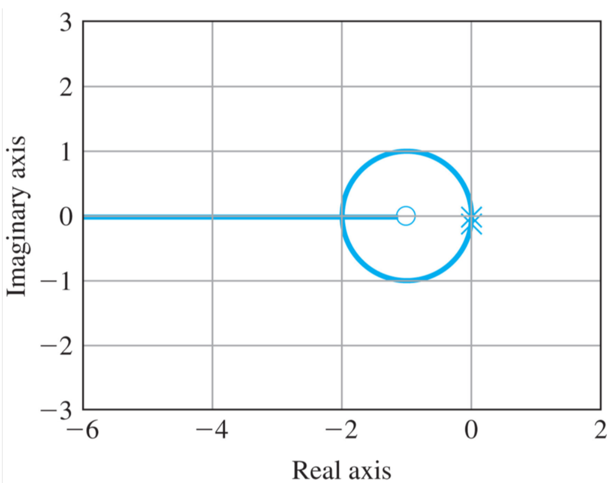
From above relation, we can know $K = -2\sigma$ and we can derive the following:

$$\sigma^2 - \omega^2 + K\sigma + K = \sigma^2 - \omega^2 - 2\sigma^2 - 2\sigma = 0 \quad \rightarrow \quad \sigma^2 + 2\sigma + \omega^2 = 0 \quad \text{for} \quad -2 < \sigma < 0$$
$$(\sigma + 1)^2 + \omega^2 = 1$$

thus we can know that the circle is plotted for $0 < K < 4$ as shown in Fig. 5.10

7. As a result,

```
s = tf('s');
sysL = (s+1)/(s^2);
rlocus(sysL)
```



- When compared P-control and PD-control for $G(s) = \frac{1}{s^2}$, the addition of the zero has pulled the locus into the LHP.

- The PD control is not practical because it requires differentiating the output variable. Thus the PD control should be modified into the filtered derivative as follow:

$$D_c(s) = k_P + k_D s \qquad \rightarrow \qquad D_c(s) = k_P + k_D \frac{s}{s/p + 1} = K \frac{s+z}{s+p}$$

  where it is called "lead compensator" provided $z < p$,

$$K = k_P + pk_D \qquad \text{and} \qquad z = \frac{pk_P}{K} = \frac{pk_P}{k_P + pk_D}$$

- (Example 5.4, Root Locus of the Satellite Control with Modified PD or Lead Compensator) Draw the root locus of the following system when the modified PD control is used ($z = 1$ and $p = 12$):

$$1 + K\frac{s+1}{s^2(s+12)} = 0$$

1. (Rule 1, Start and End)

$$\text{when } K = 0, \qquad s = 0, 0, -12 \qquad \text{poles of } L(s) = \tfrac{s+1}{s^2(s+12)}$$
$$\text{when } K = \infty, \qquad s = -1, \infty, \infty \qquad \text{zeros of } L(s)$$

2. (Rule 2, Real Axis) Negative real axis of $-12 \le s \le -1$ is locus
3. (Rule 3, Asymptotes)

$$\phi_l = \frac{180^\circ + 360(l-1)}{2} = \pm 90^\circ$$
$$\alpha = \frac{(-12)-(-1)}{2} = -\frac{11}{2} = -5.5$$

4. (Rule 4, Departure Angles and Arrival Angles)

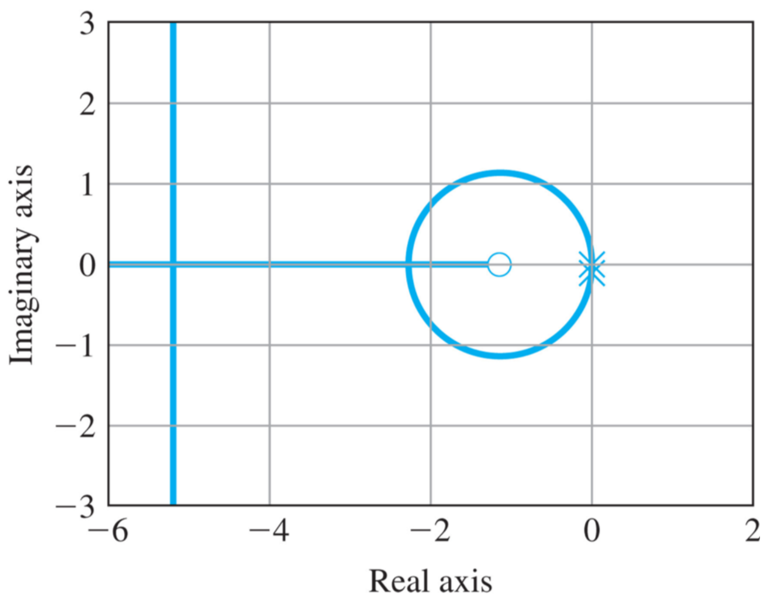
$$2\phi_{l,dep,0} = \angle(0-(-1)) - \angle(0-(-12)) - 180^\circ - 360^\circ(l-1) \qquad \rightarrow \qquad \phi_{dep,0} = \pm 90^\circ$$
$$\phi_{dep,-12} = \angle(-12-(-1)) - \angle(-12-(0)) - \angle(-12-(0)) - 180^\circ \qquad \rightarrow \qquad \phi_{dep,-12} = 0^\circ$$

5. (Rule 5, Break-in and Breakaway Points)

$$\frac{dK}{ds} = -\frac{d}{ds}\frac{s^3+12s^2}{s+1} = -\frac{(3s^2+24s)(s+1)-(s^3+12s^2)}{(s+1)^2} = -\frac{s(2s^2+15s+24)}{(s+1)^2} = 0$$

$$\rightarrow \qquad s = 0, -5.2 \ \ \textbf{breakaway} \qquad s = -2.3 \quad \textbf{break-in}$$

6. See the Fig. 5.11, the added pole $s = -12$ has been to distort the simple circle of the PD control but, for points near the origin, the locus is quite similar to the earlier case. The situation changes when the pole is brought closer in.



- (Example 5.5, Root Locus of the Satellite Control with Lead Having a Relatively Small Value for the Pole) Draw the root locus of the following system when the modified PD control is used ($z = 1$ and $p = 4$):

$$1 + K\frac{s+1}{s^2(s+4)} = 0$$

1. (Rule 1, Start and End)

$$\text{when } K = 0, \qquad s = 0, 0, -4 \qquad \text{poles of } L(s) = \tfrac{s+1}{s^2(s+4)}$$
$$\text{when } K = \infty, \qquad s = -1, \infty, \infty \qquad \text{zeros of } L(s)$$

2. (Rule 2, Real Axis) Negative real axis of $-4 \leq s \leq -1$ is locus
3. (Rule 3, Asymptotes)

$$\phi_l = \frac{180^\circ + 360(l-1)}{2} = \pm 90^\circ$$
$$\alpha = \frac{(-4) - (-1)}{2} = -\frac{3}{2} = -1.5$$

4. (Rule 4, Departure Angles and Arrival Angles)

$$2\phi_{l,dep,0} = \angle(0 - (-1)) - \angle(0 - (-4)) - 180^\circ - 360^\circ(l-1) \qquad \rightarrow \qquad \phi_{dep,0} = \pm 90^\circ$$
$$\phi_{dep,-4} = \angle(-4 - (-1)) - \angle(-4 - (0)) - \angle(-4 - (0)) - 180^\circ \qquad \rightarrow \qquad \phi_{dep,-4} = 0^\circ$$

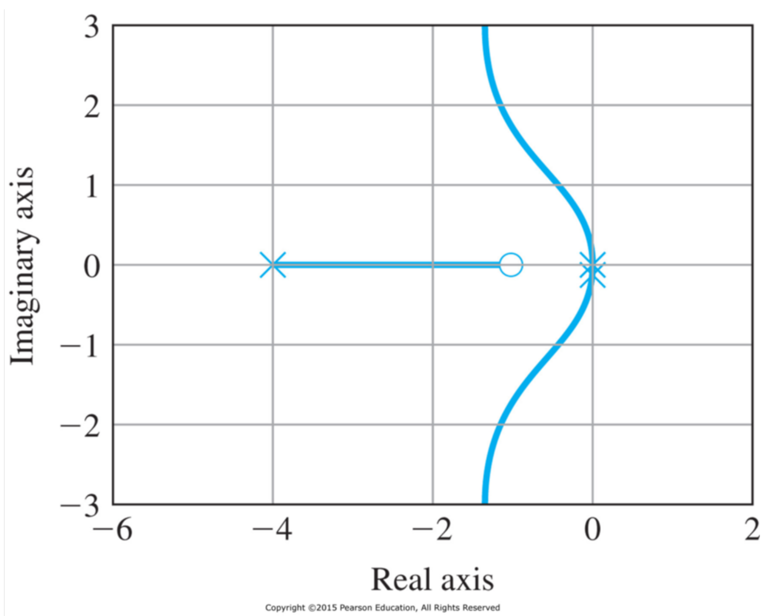
5. (Rule 5, Break-in and Breakaway Points)

$$\frac{dK}{ds} = -\frac{d}{ds}\frac{s^3+4s^2}{s+1} = -\frac{(3s^2+8s)(s+1)-(s^3+4s^2)}{(s+1)^2} = -\frac{s(2s^2+7s+8)}{(s+1)^2} = 0$$

$$\rightarrow \qquad s = 0 \ \text{ breakaway}$$

and since $\frac{-7\pm j\sqrt{15}}{4}$ are not located on the locus, it does not have any meaning

6. See the Fig. 5.12, there is no breakaway or break-in on the real axis except origin as part of the locus. A logical question might be to ask at what point they went away. As a matter of fact, it happens at $p = 9$.



- (Example 5.6, The Root Locus for the Satellite with a Transition Value for the Pole) Draw the root locus of the following system when the modified PD control is used ($z = 1$ and $p = 9$):

$$1 + K\frac{s+1}{s^2(s+9)} = 0$$

1. (Rule 1, Start and End)

$$\text{when } K = 0, \qquad s = 0, 0, -9 \qquad \text{poles of } L(s) = \tfrac{s+1}{s^2(s+9)}$$
$$\text{when } K = \infty, \qquad s = -1, \infty, \infty \qquad \text{zeros of } L(s)$$

2. (Rule 2, Real Axis) Negative real axis of $-9 \le s \le -1$ is locus
3. (Rule 3, Asymptotes)

$$\phi_l = \frac{180^\circ + 360(l-1)}{2} = \pm 90^\circ$$
$$\alpha = \frac{(-9)-(-1)}{2} = -\frac{8}{2} = -4$$

4. (Rule 4, Departure Angles and Arrival Angles)

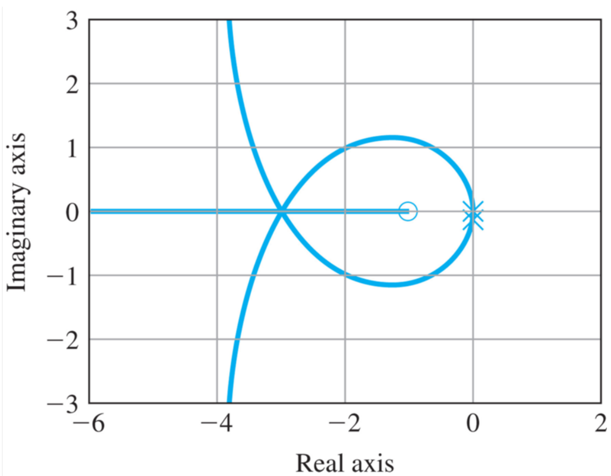
$$2\phi_{l,dep,0} = \angle(0-(-1)) - \angle(0-(-9)) - 180^\circ - 360^\circ(l-1) \qquad \rightarrow \qquad \phi_{dep,0} = \pm 90^\circ$$
$$\phi_{dep,-9} = \angle(-9-(-1)) - \angle(-9-(0)) - \angle(-9-(0)) - 180^\circ \qquad \rightarrow \qquad \phi_{dep,-9} = 0^\circ$$

5. (Rule 5, Break-in and Breakaway Points)

$$\frac{dK}{ds} = -\frac{d}{ds}\frac{s^3+9s^2}{s+1} = -\frac{(3s^2+18s)(s+1)-(s^3+9s^2)}{(s+1)^2} = -\frac{2s(s+3)^2}{(s+1)^2} = 0$$

$$\rightarrow \qquad s = 0, -3 \ \text{ breakaway} \qquad s = -3 \ \text{ break-in}$$

6. See the Fig. 5.13, the locus breaks in at $= -3$ in a triple multiple root.



- solve (Example 5.7, MATLAB) Repeat Examples 5.3, 5.4, 5.5 and 5.6 using MATLAB.
- An additional pole moving in from the far left tends to push the locus branches to the right as it approaches a given locus.
- (Example 5.8, Root Locus of the Satellite Control with a Collocated Flexibility) Plot the root locus of the characteristic equation $1 + G(s)D_c(s) = 0$, where

$$G(s) = \frac{(s+0.1)^2 + 6^2}{s^2[(s+0.1)^2 + 6.6^2]} \qquad D_c(s) = K\frac{s+1}{s+12}$$

1. (Rule 1, Start and End)

$$\text{when } K = 0, \qquad s = 0, 0, -0.1 \pm 6.6j, -12 \qquad \text{poles of } L(s) = \tfrac{(s+1)[(s+0.1)^2+6^2]}{s^2(s+12)[(s+0.1)^2+6.6^2]}$$
$$\text{when } K = \infty, \qquad s = -1, -0.1 \pm 6j, \infty, \infty \qquad \text{zeros of } L(s)$$

2. (Rule 2, Real Axis) Negative real axis of $-12 \leq s \leq -1$ is locus
3. (Rule 3, Asymptotes)

$$\phi_l = \frac{180^\circ + 360(l-1)}{2} = \pm 90^\circ$$
$$\alpha = \frac{(-12.2) - (-1.2)}{2} = -\frac{11}{2} = -5.5$$

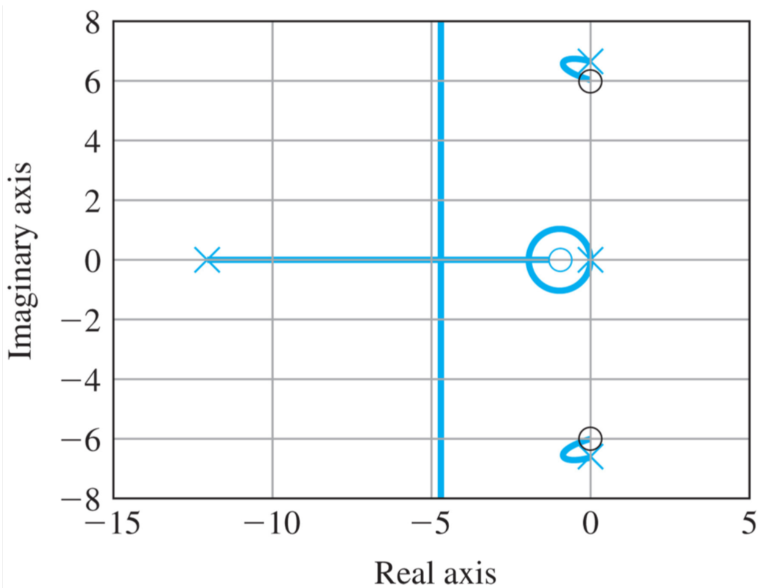
4. (Rule 4, Departure Angles and Arrival Angles)

$$
\begin{aligned}
2\phi_{l,dep,0} &= \angle(0-(-1)) + \angle(0-(-0.1+6j)) + \angle(0-(-0.1-6j)) \\
&- \angle(0-(-0.1+6.6j)) - \angle(0-(-0.1-6.6j)) - \angle(0-(-12)) \\
&- 180^\circ - 360^\circ(l-1) \qquad \rightarrow \qquad \phi_{dep,0} = \pm 90^\circ \\
\phi_{dep,-0.1+6.6j} &= \angle(-0.1+6.6j-(-1)) + \angle(-0.1+6.6j-(-0.1+6j)) + \angle(-0.1+6.6j-(-0.1-6j)) \\
&- \angle(-0.1+6.6j-0) - \angle(-0.1+6.6j-0) - \angle(-0.1+6.6j-(-0.1-6.6j)) \\
&- \angle(-0.1+6.6j-(-12)) - 180^\circ \\
&= \tan^{-1}\frac{6.6}{0.9} + 90^\circ + 90^\circ - (180^\circ - \tan^{-1}\frac{6.6}{0.1}) - (180^\circ - \tan^{-1}\frac{6.6}{0.1}) - 90^\circ - \tan^{-1}\frac{6.6}{11.9} - 180^\circ \\
&= 82.2^\circ + 180^\circ - (180^\circ - 89.1^\circ) - (180^\circ - 89.1^\circ) - 90^\circ - 29^\circ - 180^\circ = -217.7^\circ = +142.3^\circ \\
\phi_{dep,-0.1-6.6j} &= -142.3^\circ \\
\psi_{arr,-0.1+6j} &= \angle(-0.1+6j-(-0.1+6.6j)) + \angle(-0.1+6j-(-0.1-6.6j)) + \angle(-0.1+6j-(0)) \\
&+ \angle(-0.1+6j-(-0)) + \angle(-0.1+6j-(-12)) - \angle(-0.1+6j-(-0.1-6j)) \\
&- \angle(-0.1+6j-(-1)) + 180^\circ \\
&= -90^\circ + 90^\circ + 2(180^\circ - \tan^{-1}\frac{6}{0.1}) + \tan^{-1}\frac{6}{11.9} - 90^\circ - \tan^{-1}\frac{6}{0.9} + 180^\circ \\
&= 182^\circ + 26.7^\circ - 90^\circ - 81.5^\circ + 180^\circ = 217.2^\circ \\
\psi_{arr,-0.1-6j} &= -217.2^\circ
\end{aligned}
$$

5. (Rule 5, Break-in and Breakaway Points) too complex although they exist some points satisfying

$$\frac{dK}{ds} = 0$$

6. See the Fig. 5.16. In the collocated case, the presence of a single flexible mode introduces a lightly damped root to the characteristic equation but does not cause the system to be unstable.



- solve (Example 5.9)
- (Example, Root Locus for a Non-collocated Case) Plot the root locus of the characteristic equation $1 + G(s)D_c(s) = 0$, where

$$G(s) = \frac{1}{s^2[(s+0.1)^2 + 6.6^2]} \qquad D_c(s) = K\frac{s+1}{s+12}$$

1. (Rule 1, Start and End)

$$\text{when } K = 0, \quad s = 0, 0, -0.1 \pm 6.6j, -12 \quad \text{poles of } L(s) = \tfrac{s+1}{s^2(s+12)[(s+0.1)^2+6.6^2]}$$
$$\text{when } K = \infty, \quad s = -1, \infty, \infty, \infty, \infty \quad \text{zeros of } L(s)$$

2. (Rule 2, Real Axis) Negative real axis of $-12 \le s \le -1$ is locus
3. (Rule 3, Asymptotes)

$$\phi_l = \frac{180^\circ + 360(l-1)}{4} = \pm 45^\circ, \pm 135^\circ \qquad \alpha = \frac{(-12.2) - (-1)}{4} = -\frac{11.2}{4} = -2.8$$

4. (Rule 4, Departure Angles and Arrival Angles)

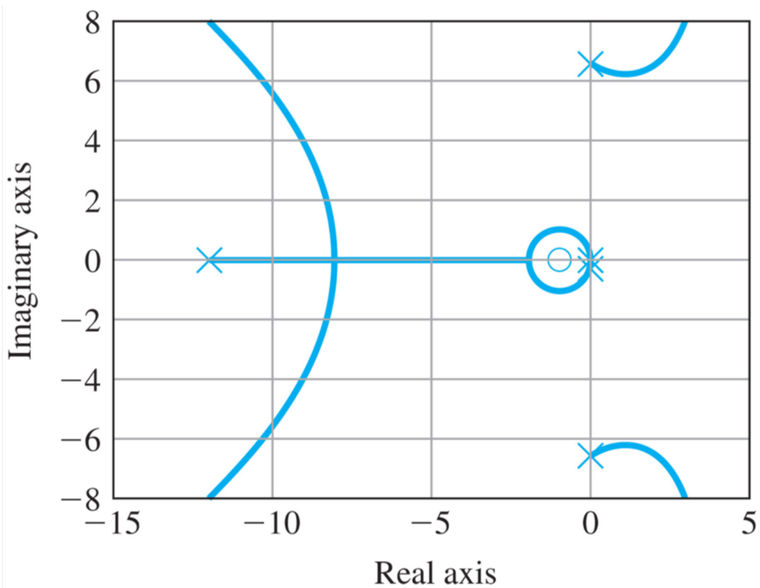
$$\begin{aligned}
\phi_{dep,-0.1+6.6j} &= \angle(-0.1 + 6.6j - (-1)) - \angle(-0.1 + 6.6j - 0) - \angle(-0.1 + 6.6j - 0) - \angle(-0.1 + 6.6j - (-0.1 - 6.6
\\
&- \angle(-0.1 + 6.6j - (-12)) - 180^\circ \\
&= \tan^{-1}\frac{6.6}{0.9} - (180^\circ - \tan^{-1}\frac{6.6}{0.1}) - (180^\circ - \tan^{-1}\frac{6.6}{0.1}) - 90^\circ - \tan^{-1}\frac{6.6}{11.9} - 180^\circ \\
&= 82.2^\circ - (180^\circ - 89.1^\circ) - (180^\circ - 89.1^\circ) - 90^\circ - 29^\circ - 180^\circ = -37.7^\circ
\end{aligned}$$

$$\phi_{dep,-0.1-6.6j} = 37.7^\circ$$

5. (Rule 5, Break-in and Breakaway Points) too complex although they exist some points satisfying

$$\frac{dK}{ds} = 0$$

6. See the figure. In the non-collocated case, the root leaves pole down and to the right, toward the unstable region. We would expect the system to soon become unstable as gain is increased.



- solve (Example 5.10)
- (Example, Root Locus Having Complex Multiple Roots) Plot the root locus of the characteristic equation $1 + KL(s) = 0$, where

$$L(s) = \frac{1}{s(s+2)[(s+1)^2+4]}$$

1. (Rule 1, Start and End)

$$\text{when } K = 0, \qquad s = 0, -2, -1 \pm 2j \qquad \text{poles of } L(s)$$
$$\text{when } K = \infty, \qquad s = \infty, \infty, \infty, \infty \qquad \text{zeros of } L(s)$$

2. (Rule 2, Real Axis) Negative real axis of $-2 \le s \le 0$ is locus
3. (Rule 3, Asymptotes)

$$\phi_l = \frac{180^\circ + 360(l-1)}{4} = \pm 45^\circ, \pm 135^\circ$$
$$\alpha = \frac{(-4)-(0)}{4} = -1$$

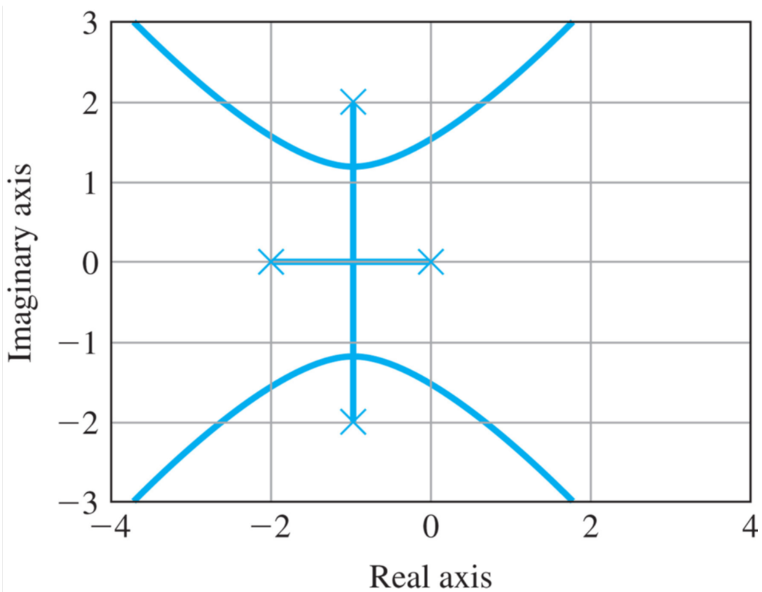
4. (Rule 4, Departure Angles and Arrival Angles)

$$\begin{aligned}
\phi_{dep,-1+2j} &= -\angle(-1+2j-0) - \angle(-1+2j-(-2)) - \angle(-1+2j-(-1-2j)) - 180^\circ \\
&= -(180^\circ - \tan^{-1} 2) - \tan^{-1} 2 - 90^\circ - 180^\circ \\
&= -116.6^\circ - 63.4^\circ - 90^\circ - 180^\circ = -90^\circ \\
\phi_{dep,-1-2j} &= 90^\circ
\end{aligned}$$

5. (Rule 5, Break-in and Breakaway Points) too complex although they exist some points satisfying

$$\frac{dK}{ds} = -(s+1)(2s^2+4s+5)0 \qquad \rightarrow \qquad s = -1, -1 \pm 1.22j \quad \textbf{breakaway}$$

6. See the figure



1. (Homework #5, Due : June 19, 2020) Solve and Submit 5.1, 5.2, 5.3, 5.9, 5.13, 5.20, 5.24, 5.28, 5.35, 5.37